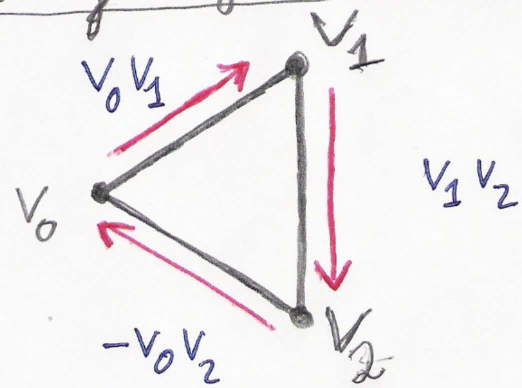
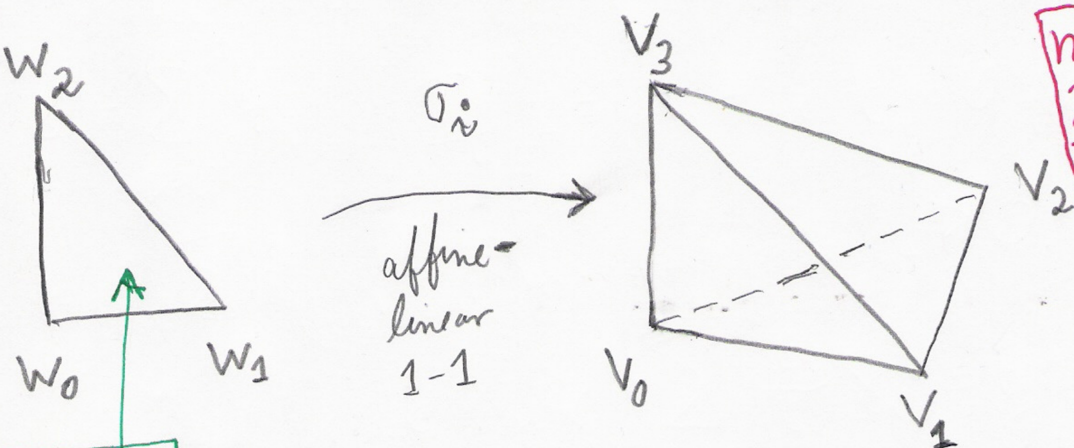
Orienting edges of a triangle = 2-simplex



Orientations for the faces of a tetrahedron = 3-simplex



$$\boxed{\text{normal} = \frac{\partial \sigma_i}{\partial x} \times \frac{\partial \sigma_i}{\partial y}}$$

all (x,y) s.t. $0 \le x,y \le 1$, $x+y \le 1$.

$\sigma_0(W_0W_1W_2) = V_1V_2V_3$ normal $(1,1,1)$ OUT

$\sigma_1(W_0W_1W_2) = V_0V_2V_3$ normal $(1,0,0)$ IN

$\sigma_2(W_0W_1W_2) = V_0V_1V_3$ normal $(0,-1,0)$ OUT

$\sigma_3(W_0W_1W_2) = V_0V_1V_2$ normal $(0,0,1)$ IN

σ_i maps the 2-simplex into the face of the 3-simplex opposite the i-th vertex, sending (x,y) to $xV_a + yV_b + (1-x-y)V_c$, where $a < b < c$.

Therefore, to get the outward normal on all four faces of the tetrahedron's boundary, we should take the chain

$$V_1 V_2 V_3 - V_0 V_2 V_3 + V_0 V_1 V_3 - V_0 V_1 V_2 =$$

$$\partial_0 - \partial_1 + \partial_2 - \partial_3$$

where ∂_i = face opposite the i^{th} vertex.

In the 2D case we got the analogous formula $\partial_0 - \partial_1 + \partial_2$.

So an educated guess for the right orientation for the boundary of an n-simplex is

$$\sum_{i=0}^{n} (-1)^i \partial_i .$$

FACE IDENTITIES

$0 \leq i < j$

$\partial_i \circ \partial_j$ = remove vertex v_j, then remove v_i.

One gets the same result from the composite

$$\partial_{j-1} \circ \partial_i .$$

$$d_{n-1} \circ d_n = \left(\sum (-1)^i \partial_i\right) \circ \left(\sum (-1)^j \partial_j\right) =$$

$$\sum_{i,j} (-1)^{i+j} \partial_i \circ \partial_j = \sum_{i<j} (-1)^{i+j} \partial_i \partial_j +$$

$\sum_{i \geq j} (-1)^{i+j} \partial_i \partial_j$. Apply the Face Identities to the first sum, getting

$$\sum_{i<j}(-1)^{i+j}\partial_{j-1}\partial_i + \sum_{i\geq j}(-1)^{i+j}\partial_i\partial_j .$$

Do a change of variables for the first term, with $u=j-1$ $v=i$. Then we get

$$\sum_{u\geq v}(-1)^{u+v-1}\partial_u\partial_v + \sum_{i\geq j}(-1)^{i+j}\partial_i\partial_j .$$

Notice that the first and second terms cancel each other, so that

$$\boxed{d_{n-1}\circ d_n = 0.}$$