COMPUTING SIMPLICIAL HOMOLOGY

Setting (C_*, d_*) is a chain complex, and each chain group C_q is finitely generated and free abelian. Let $r_q = \text{rank } C_q$. Suppose we are given ordered free bases $\mathcal{B}_q$ of the C_q and that we have explicit formulas for the boundary map $d_q : C_q \to C_{q-1}$. In other words, given a free basis element $x \in \mathcal{B}_q$, we know how to express dx explicitly as a $\mathbb{Z}$-linear combination of the basis elements in $\mathcal{B}_{q-1}$.

All of these conditions hold for the simplicial chain complex of a simplicial complex (P, K) with linearly ordered vertices.

Preliminary observations. ① The image of d_q is a finitely generated free abelian group $A_q \subseteq C_{q-1}$. Let a_q denote its rank.

② If $Z_q = \text{Kernel } d_q$, then since A_q is free we have an isomorphism $Z_q \oplus A_q \cong C_q$.

[Since C_q maps <u>onto</u> A_q via d_q.]
Let $z_q = \operatorname{rank} Z_q$, so that $r_q = z_q + a_q$.

<u>Matrix reduction</u> The data $\{d_{q+1}, \mathcal{B}_{q+1}, \mathcal{B}_q\}$ determine an $r_q \times r_{q+1}$ matrix over the integers which we shall call P. By fundamental results on matrices over a principal ideal domain, there are invertible matrices S_1 and S_2 such that $P_1 = S_1 P S_2$ is "diagonal" (S_1 and S_2 are invertible over $\mathbb{Z}$). This translates into the existence of ordered free bases $\mathfrak{W}_{q+1}$ and $\mathfrak{W}_q'$ for C_{q+1} and C_q such that

$$d_{q+1} x_i = y_i \quad i = 1, \dots, s$$

$$d_{q+1} x_i = m_i y_i \quad i = s+1, \dots, r_{q+1} \qquad m_i > 1 \text{ all } i$$

$$d_{q+1} x_i = 0 \quad i > r_{q+1}$$

Then y_i ($i \le s$) and $m_i y_i$ ($s < i \le a_q$) are a free set of generators for A_{q+1}

<u>NOTE.</u> Any of the subsets of x_i's might be empty!

Let $\widetilde{A}_{q+1}$ = subgroup gen. by $y_1, \ldots, y_{r_{q+1}}$, then A_{q+1} has finite index in $\widetilde{A}_q$. There is a natural splitting $C_q = \widetilde{A}_{q+1} \oplus Y_q$

gen by remaining y_i's (if any)

Since $d_q | A_q = 0$ and the codomain of d_q is torsion free, we also have $d_q | \widetilde{A}_{q+1} = 0$. In other words, $\widetilde{A}_{q+1} \subseteq Z_q$.

<u>CLAIM</u> We have $Z_q \cong \widetilde{A}_{q+1} \oplus X_q$ for some direct summand X_q of Y_q.

<u>PROOF.</u> Look at the composite

$$Y_q \subseteq C_q \xrightarrow{\text{ONTO}} A_q.$$

Since $\widetilde{A}_{q+1}$ goes to zero in $A_q \subseteq C_{q-1}$ and $C_q = \widetilde{A}_{q+1} \oplus Y_q$, the composite must be onto, so we have $Y_q \cong A_q \oplus X_q$ for some X_q by freeness of A_q. It follows immediately that $Z_q = \widetilde{A}_{q+1} \oplus X_q$.

<u>Note</u> The rank of X_q is $z_q - a_{q+1} =$ $r_q - a_q - a_{q+1}$.

We can now give an explicit formula.

<u>Theorem</u> $H_q(C) \cong Z_q / A_{q+1} \cong$

$$X_q \oplus \left[\tilde{A}_{q+1} / A_{q+1}\right] \cong$$

$$\mathbb{Z}^{r_q - a_q - a_{q+1}} \oplus \underbrace{\mathbb{Z}_{c_{s+1}} \oplus \cdots \oplus \mathbb{Z}_{c_{q_r}}}_{\text{"torsion subgroup"}}.$$

$r_q - a_q - a_{q+1}$ is often called the q^{th} <u>Betti number</u> and the c_i's are then called <u>torsion coefficients</u>. The integral matrix reduction theorems imply that we can carry out the process so that $c_{s+1} | c_{s+2}, \ldots, c_{s+j} | c_{s+j+1} \ldots$

For these choices, one has uniqueness:

If $\mathbb{Z}_{k_1} \oplus \cdots \oplus \mathbb{Z}_{k_u} \cong \mathbb{Z}_{l_1} \oplus \cdots \oplus \mathbb{Z}_{l_v}$ s.t.

$1 < k_1 | k_2 \ldots | k_u \quad 1 < l_1 | l_2 \ldots | l_v$,

then $u = v$ and $k_i = l_i \ \forall i$.