NAME: Answer Key

Mathematics 205C, Spring 2011, Examination 1

Work all questions, and unless indicated otherwise give reasons for your answers. The point values for individual problems are indicated in brackets.

Unless explicitly stated otherwise, all topological spaces are assumed to be Hausdorff and locally arcwise connected.

#	SCORE
1	
2	
3	
4	
TOTAL	

1. [*30 points*] (*i*) Let (X, x_0) and (Y, y_0) be arcwise connected spaces that are locally simply connected, and assume further that Y is simply connected. Suppose that $p : (W, w_0) \to \big(X \times Y, (x_0, y_0) \big)$ is a covering space projection such that W is also arcwise connected. Prove that W is homeomorphic to $V \times Y$, where $q : (V, v_0) \to (X, x_0)$ is a covering space projection such that V is arcwise connected. [*Hint:* If $q : V \to X$ is a covering with V arcwise connected, what can we say about $q \times \mathrm{id}_Y : V \times Y \to X \times Y$? Why is this so?]

(*ii*) If $n \geq 2$, determine the number of equivalence classes of covering space structures over $\mathbf{RP}^n \times \mathbf{RP}^n$, where $\mathbf{RP}^n$ is real projective n-space.

(i) Connected covering spaces over X are classified up to equivalence by subgroups of $\pi_1(X, x_0)$, and likewise connected covering spaces over Y are classified by subgroups of $\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0)$ [since Y is simply connected]. Suppose that $p : W \to X \times Y$ corresponds to $H \subseteq \pi_1(X \times Y) \cong \pi_1(X)$. Let $q : V \to X$ also correspond to H. CLAIM $V \times Y \to X \times Y$ is a (connected) covering space projection; if $U \subseteq X$ is open and evenly covered, then the same holds for $U \times Y \subseteq X \times Y$. Now $V \times Y \to X \times Y$ corresponds to H by construction, so by the classification of covering space projections we have $W \cong X \times Y$.

(ii) $\pi_1(\mathbb{RP}^n \times \mathbb{RP}^n) = \mathbb{Z}_2 \times \mathbb{Z}_2$ which has ~~four~~ five subgroups $\{1\}$, $\mathbb{Z}_2 \times \{0\}$, $\{0\} \times \mathbb{Z}_2$, diagonal $(\mathbb{Z}_2)$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$.

2. [*20 points*] Suppose that (X, x_0) and (Y, y_0) are locally simply connected and arcwise connected, let $f : (X, x_0) \to (Y, y_0)$ be a continuous mapping, and let

$$p : (\widetilde{X}, \xi_0) \longrightarrow (X, x_0) \quad \text{and} \quad q : (\widetilde{Y}, \eta_0) \longrightarrow (Y, y_0)$$

be universal covering space projections. Prove that there is a unique continuous mapping $F : (\widetilde{X}, \xi_0) \to (\widetilde{Y}, \eta_0)$ such that $q\,F = f\,p$. [*Hint:* Use the Lifting Criterion.]

Consider

$$\begin{tikzcd} & & (\tilde{Y}_0, \eta_0) \arrow[d, "q"] \\ (\tilde{X}, \xi_0) \arrow[r, "p"] \arrow[urr, dashed, "F?"] & (X_0, x_0) \arrow[r, "f"] & (Y_0, y_0) \end{tikzcd}$$

A lifting F exists $\Longrightarrow f_* p_* [\pi_1(\tilde{X})] \subseteq \pi_1(\tilde{Y})$.
Since $\pi_1(\tilde{X})$ and $\pi_1(\tilde{Y})$ are trivial, the criterion holds and hence a lifting exists. By the connectness and Hausdorff assumptions, F is unique (for such cases, a lifting is always unique if it exists).

3. [*25 points*] Suppose that X is an arcwise connected space such that $X = U \cup V$, where U and V are open and arcwise connected and their intersection $U \cap V$ is also arcwise connected. Let $p \in U \cap V$.

(i) Show that if U is simply connected and $\pi_1(V, p)$ is abelian, then $\pi_1(X)$ is also abelian.

(ii) Explain why the same conclusion does not hold if we merely assume that $\pi_1(U, p)$ is abelian. It will suffice to give a counterexample.

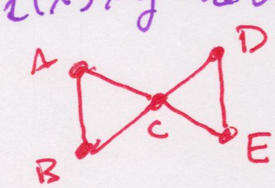
(i) $\pi_1(X)$ is the push out of

$$\begin{tikzcd} \pi_1(U \cap V) \arrow[r] \arrow[d] & \pi_1(U) \\ \pi_1(V) & \end{tikzcd}$$

So it suffices to show that in such a pushout $\pi_1(X)$ is a quotient of $\pi_1(V)$ [a quotient of an abelian group is abelian]. But we have

$$\begin{tikzcd} K \arrow[r] \arrow[d] & \{1\} \arrow[d] \\ H \arrow[r] & \pi_1(X) \end{tikzcd}$$

and in particular $\pi_1(X)$ is generated by H and hence is a quotient of H. Since H is abelian, so is $\pi_1(X)$ by the discussion above.

(ii) $X =$ 

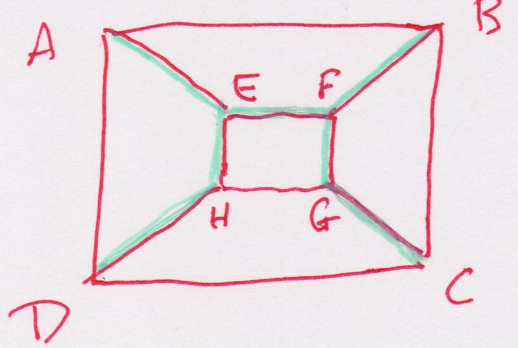
$U = X -$ midpt DE
$V = X -$ midpt AB.

Then $\pi_1(U) \cong \pi_1(V) \cong \mathbb{Z}$, but $\pi_1(X)$ is free on two generators and hence nonabelian.

4. [*25 points*] The graph $(X, \mathcal{E})$ determined by the edges of a standard cube can be presented as a graph with vertices A, B, C, D, E, F, G, H and edges

$$AB,\ BC,\ CD,\ AD, \qquad EF,\ FG,\ GH,\ EH, \qquad AE,\ BF,\ CG,\ DH\ .$$

Find the nonnegative integer m such that $\pi_1(X, \mathrm{pt.})$ is a free group on m generators, and find a maximal tree in $(X, \mathcal{E})$. [*Hint:* Making a drawing of the graph may be extremely useful. An additional sheet of paper is provided.]



$m = 1 - \chi(X, \mathcal{E})$. $(X, \mathcal{E})$ has 8 vertices & 12 edges, so

$\chi(X, \mathcal{E}) = 8 - 12 = -4$ and $m = 1 - (-4) = \underline{\underline{5 \text{ gens.}}}$

If T is a maximal tree, then T has 8 vertices and the number K of edges is given by $\chi(T) = 1 = 8 - K$, so $K = 7$. The subgraph outlined in green is one maximal tree (there are also others!)

Notice this has 7 edges.