Mathematics 205C, Spring 2011, Final Examination

Solutions

1. [*25 points*] Let $(X, \mathcal{E})$ be a connected graph, let $n \geq 2$ be an integer, and let $(X', \mathcal{E}')$ be a connected n-sheeted covering with the canonical graph structure. Suppose that $\pi_1(X)$ is a free group on $g \geq 1$ generators. Then $\pi_1(X')$ is a free group on $h \geq 1$ generators for some positive integer h. Derive a formula for h in terms of g and n. [*Hint:* Look at the Euler characteristics of X and X'.]

SOLUTION

The Euler characteristics of X and X' are given by $\chi(X) = 1 - g$ and $\chi(X') = 1 - h$. Since X' is an n-sheeted covering of X we have $\chi(X') = n\,\chi(X)$. Therefore

$$h \quad = \quad 1 - \chi(X') \quad = \quad 1 - n\,\chi(X) \quad = \quad 1 - n\left(1 - g\right) \quad = \quad ng - n + 1\ .$$

2. [*25 points*] Let A_* and B_* be chain complexes, and suppose that $f : A_* \to B_*$ is a map of chain complexes. Prove that for each integer q there is a well-defined homomorphism of homology groups $f_* : H_q(A) \to H_q(B)$ such that if $u \in H_q(A)$ is represented by $x \in A_q$, then $f_*(u)$ is represented by $f(x) \in B_q$.

SOLUTION

First of all we need to prove that if $dx = 0$ then $df(x) = 0$; but $dx = 0$ and the fact that f is a chain map imply

$$0 \quad = \quad f(dx) \quad = \quad df(x)$$

so this is true. Next, suppose that x and y represent u, so that $x - y = dz$ for some z. Then

$$f(y) \quad = \quad f(x - dz) \quad = \quad f(x) - f(dz) \quad = \quad f(x) - df(z)$$

so that $f(x)$ and $f(y)$ represent the same element in $H_q(B)$. To see the map is a homomorphism, note that if x_i represents u_i for $i = 1, 2$, then $f_*(u_1 + u_2)$ is represented by

$$f(x_1 + x_2) \quad = \quad f(x_1) + f(x_2)$$

and by the basic condition in the problem the right hand side represents $f_*(u_1) + f_*(u_2)$.

3. [*25 points*] If $k \geq 1$ then the half-space $\mathbb{R}^k_+$ is defined to be the set of all points in $\mathbb{R}^k$ whose first coordinates are nonnegative. Prove that if $m \neq n$ are positive integers then $\mathbb{R}^m_+$ and $\mathbb{R}^n_+$ are not homeomorphic. [*Hint:* $\mathbb{R}^k_+$ has an open dense subset which is open in $\mathbb{R}^k$; namely, the set of all points whose first coordinates are positive. If X is homeomorphic to both of these spaces, why does X have open dense sets homeomorphic to $\mathbb{R}^m$ and $\mathbb{R}^n$? Recall that the intersection of two such open dense subsets is always dense and hence nonempty.]

SOLUTION

Suppose there is a homeomorphism $f : \mathbb{R}^m_+ \to \mathbb{R}^n_+$. Let $V \subset \mathbb{R}^n_+$ be the open subset (even in $\mathbb{R}^n$) of all points whose first coordinates are positive, let $U_0 \subset \mathbb{R}^m_+$ be the open subset (even in $\mathbb{R}^m$) of all points whose first coordinates are positive, and let $U = f[U_0]$. Since f is a homeomorphism we have $U = \mathbb{R}^n_+ - \left[\mathbb{R}^{n-1} \times \{0\}\right]$, and since f is a homeomorphism the deleted set is closed in $\mathbb{R}^n_+$, so that U is also open in $\mathbb{R}^n_+$. Clearly U and V are both dense subsets (every point in $\mathbb{R}^k_+$ is a limit of points whose last coordinates are positive), and since the intersection of two open dense subsets is open and dense we conclude that $U \cap V$ is also dense in $\mathbb{R}^n_+$. Now $U \cap V$ is open in both U and V, where one of the latter is homeomorphic to an open subset of $\mathbb{R}^m$ and the other is homeomorphic to an open subset of $\mathbb{R}^n$. By Invariance of Dimension, this implies that $m = n$.

4. [*25 points*] (*a*) Let Δ_3 be the standard 3-dimensional simplex with vertices $\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$. Write down the explicit formula for the boundary of the generator $\mathbf{e}_0\,\mathbf{e}_1\,\mathbf{e}_2\,\mathbf{e}_3$ in the simplicial chain complex $C_*(\Delta_3)$ as a linear combination of the generators for $C_2(\Delta_3)$.

(*b*) Suppose that $(P, \mathbf{K})$ is a connected simplicial complex, and let $A, B \in \mathbf{K}$. Prove that there is a sequence of simplices $A = S_0, \cdots, S_p = B$ such that each intersection is nonempty. [*Hint:* Define a binary relation on simplices such that A and B are related if a chain of the given type exists. Why is this an equivalence relation, and why is the union of all simplices in a given equivalence class a closed and open subset of P?]

SOLUTION

(*a*) The boundary is given by

$$\mathbf{e}_1\,\mathbf{e}_2\,\mathbf{e}_3 \;-\; \mathbf{e}_0\,\mathbf{e}_2\,\mathbf{e}_3 \;+\; \mathbf{e}_0\,\mathbf{e}_1\,\mathbf{e}_3 \;-\; \mathbf{e}_0\,\mathbf{e}_1\,\mathbf{e}_2\ .$$

(*b*) The binary relation on simplices is just the equivalence relation generated by the binary relation $A \sim B$ if and only if $A \cap B$ is nonempty. Given a simplex A, let P_A be the set of all points in simplices which belong to the equivalence class of A, so that P_A is the union of all simplices in this class. Clearly P is the union of the sets P_A, while $A \sim B$ implies $P_A = P_B$ and if $P_A \cap P_B \neq \emptyset$, then a point x in this intersection must lie on some simplex C which is in the same equivalence class as A and likewise for B; in other words the subsets P_A and P_B are either disjoint or identical. Each of these is a finite union of simplices and hence closed, so we have a decomposition of P into finitely many pairwise disjoint closed subsets. Since P is connected there can be only one subset in this decomposition, so $P = P_C$ for every simplex C, and hence for any two simplices we have $A \sim B$.

5. [*25 points*] (a) Suppose that X is a connected and locally simply connected space such that $\pi_1(X)$ is finite of odd order. Prove that X has no connected 2-sheeted covering spaces.

(b) Suppose that $X = U \cup V$ where U, V and $U \cap V$ are all open and arcwise connected. Prove that $\pi_1(X)$ is finitely generated if both $\pi_1(U)$ and $\pi_1(V)$ are.

SOLUTION

(a) The equivalence classes of connected coverings are in 1–1 correspondence with subgroups of $\pi_1(X)$, and if a covering space corresponds to the subgroup H then the number of sheets is equal to the index of H in $\pi_1(X)$. If the latter has odd order, then by Lagrange's Theorem on cosets the index of H must also be odd. In particular, it cannot be equal to 2.

(b) By the Seifert-van Kampen Theorem we know that $\pi_1(X)$ is generated by the images of $\pi_1(U)$ and $\pi_1(V)$. If the latter have finite generating sets A and B then the union of their images in $\pi_1(X)$ will be a finite generating set for $\pi_1(X)$.

6. [*25 points*] Let $(P, \mathbf{K})$ be an n-dimensional simplicial complex. Prove that $H_n(P, \mathbf{K})$ is a finitely generated free abelian group. [*Hint:* Why is $H_n(P, \mathbf{K})$ isomorphic to the kernel of d_n?]

SOLUTION

By definition the homology group is isomorphic to the kernel of D_n modulo the image of d_{n+1}. Since the complex is n-dimensional, the domain of the latter is zero and hence this map is zero, and therefore $H_n(P, \mathbf{K})$ is isomorphic to the kernel of d_n. Since this kernel is a subgroup of the finitely generated abelian group $C_n(P, \mathbf{K})$, it follows that the kernel must be a finitely generated free abelian group. In particular, it has no nonzero elements of finite order.

7. [*25 points*] Given a space X and $x \in X$, the local homology groups of X at x are defined by $H_*(X, X - \{x\})$. Prove that if U is an open neighborhood of x then these groups only depend on U (and its subspace topology). [*Note:* Recall the default assumption that X is Hausdorff.]

SOLUTION

We know that $X = X - \{x\} \cup U$ and $U - \{x\} = X - \{x\} \cap U$, so by excision the morphism from $H_*(U, U - \{x\})$ to $H_*(X, X - \{x\})$ induced by inclusion will be an isomorphism.

8. [*25 points*] Suppose that $X = U \cup V$ where U and V open in X. Assume further that the maps $H_q(U \cap V)$ to $H_q(U)$ induced by the inclusion $U \cap V \subset U$ are isomorphisms (in all dimensions) and the maps $H_q(U \cap V)$ to $H_q(V)$ induced by the inclusion $U \cap V \subset V$ are zero (in all dimensions). Prove that the maps $H_q(V)$ to $H_q(X)$ induced by the inclusion $V \subset X$ are also isomorphisms (in all dimensions).

Extra credit. [*20 points*] Prove the same conclusion holds without the restriction on the maps $H_q(U \cap V)$ to $H_q(V)$. [*Hint:* If $f : A \to B$ is a homomorphism of abelian groups and $F : A \to A \oplus B$ is the graph map $F(a) = (a, f(a))$, show that the quotient of $A \oplus B$ modulo the image of F is isomorphic to B by considering the map $\varphi : A \oplus B \to B$ with $\varphi(a, b) = b - f(a)$.]

SOLUTION

Consider the Mayer-Vietoris sequence for (U, V). By assumption the maps from $H_*(U \cap V)$ to $H_*(U) \oplus H_*(V)$ are 1–1 because this is true for their projections onto $H_*(U)$ are in fact isomorphisms. This means that the boundary maps

$$\Delta : H_{q+1}(U \cup V) \to H_q(U \cap V)$$

are all zero and the maps from $H_(U) \oplus H_*(V)$ to $H_*(U \cup V)$ are onto. By exactness we know that the kernel of this onto map is the image of the previous map, which is just $H_*(U) \oplus 0$. But this means that the map from $H_*(V)$ to $H_*(U \cup V)$ must be an isomorphism.

EXTRA CREDIT. In this case we still know that the map from $H_*(U \cap V)$ to direct sum $H_*(U) \oplus H_*(V)$ is 1–1 and hence Δ is also zero in this case, so that the Mayer-Vietoris sequence becomes a collection of short exact sequences:

$$0 \to H_q(U \cap V) \to H_q(U) \oplus H_q(V) \to H_q(U \cup V) \to 0$$

If we identify $H_q(U \cap V)$ with $H_q(U) = A$ (by definition) using the isomorphism assumption and let $H_q(V) = B$, then the short exact sequence imply that $H_q(U \cup V)$ is the quotient of $A \oplus B$ modulo the image C of a homomorphism $A \to A \oplus B$ sending a to $(a, g(a))$ for some homomorphism $g : A \to B$. Therefore it suffices to show that the quotient of $A \oplus B$ by such a subgroup is isomorphic to B. The simplest way to do this is to define a homomorphism from $A \oplus B$ to B which is onto and whose kernel is C, and this can be done by defining $h : A \oplus B \to B$ so that $h(a, b) = b - g(a)$.

An extra page for use if needed