EXERCISES FOR MATHEMATICS 205C

SPRING 2011

File Number 01

DEFAULT HYPOTHESES. Unless specifically stated otherwise, all spaces are assumed to be Hausdorff and locally arcwise connected.

1. Let $p : (X, x_0) \to (Y, y_0)$ be a covering space projection with X and Y both (arcwise) connected. Suppose that $h : (Y, y_0) \to (Y, y_0)$ is a homeomorphism. Prove that the composite $h \circ p$ is also a base point preserving covering space projection, and if the original covering space data correspond to the subgroup $H \subset \pi_1(Y, y_0)$, then the new covering space data correspond to $h_*[H]$.

2. Let $e = (1, 1) \in T^2$, and let $A \subset \pi_1(T^2, e)$ be the subgroup generated by $(a, 0)$ and $(0, b)$ where a and b are positive integers, and let $p : (X, x_0) \to (T^2, e)$ denote the covering space associated to A. Prove that X is homeomorphic to T^2.

3. Suppose we are given a covering space projection $p : (X, x_0) \to (Y, y_0)$ such that X is simply connected (hence Y is connected). Assume further that there are base point preserving mappings $F : X \to X$ and $f : Y \to Y$ such that $f \circ p = p \circ F$.

(i) Let $a \in \pi_1(Y, y_0)$. Prove that there is a unique $\varphi(a) \in \pi_1(Y, y_0)$ such that $f(x_0 a) = x_0 \varphi(a)$.

(ii) Prove that φ is equal to the homomorphism f_* from $\pi_1(Y, y_0)$ to itself.

(iii) If $f : (T^n, 1) \to (T^n, 1)$ is continuous, then under the isomorphism $\pi_1(T^n, 1) \cong \mathbb{Z}^n$ the map f_* corresponds to an $n \times n$ matrix A with integral matrices. Prove that, conversely, every such matrix is realized by a suitable mapping f. Furthermore, prove that A can be realized by a homeomorphism if and only if its determinant is equal to ± 1. [*Hint:* Let $\Phi : \mathbb{R}^n \to T^n$ send $(t_1, \cdots, t_n)$ to $(\exp 2\pi i\, t_1, \cdots, \exp 2\pi i\, t_n)$, and let F be the linear transformation of $\mathbb{R}^n$ defined by the integral matrix A. Why does F send $\mathbb{Z}^n$ to itself, and why does this imply that F passes to a basepoint preserving self-map of T^n? Use Exercise 1 to show that f_* corresponds to multiplication by A.]

4. Let $n \geq 2$ and let $p : (X, x_0) \to (\mathbb{RP}^n \times \mathbb{RP}^n, y_0)$ be a covering space projection wuch that X is connected.

(i) Prove that either p is a homeomorphism or else the covering has a (finite) even number of sheets.

(ii) Determine the number of equivalence classes of connected covering spaces over $\mathbb{RP}^n \times \mathbb{RP}^n$.

5. Let $p : (X, x_0) \to (Y, y_0)$ be a finite covering space projection where X is connected and both X and Y are locally simply connected. Prove that there is a covering space projection $q : (W, w_0) \to (X, x_0)$ with the following properties:

(1) The space W is connected.

(2) The map q is a regular covering space projection with finitely many sheets.

(3) The map q has a factorization $q = p \circ q'$ for some mapping $q' : W \to X$. [*Hint:* If $\widetilde{Y}$ is the universal covering space of Y, then $X \cong \widetilde{Y}/\pi_1(X, x_0)$, and a similar statement holds for W.]

(4) ($\star$) The map q' is also a regular covering space projection with finitely many sheets.

[*Hint:* This is basically a result about groups with subgroups of finite index.]

6. The **Klein bottle** K can be constructed as a quotient of T^2 modulo the equivalence relation determined by identifying (z_1, z_2) with $(-z_1, \overline{z_2})$, where $\overline{z}$ denotes the complex conjugate of z. If we define a free $G = \mathbb{Z}_2$ action on T^2 by $g \cdot (z, w) = (-z, \overline{w})$ where $1 \neq g \in G$, then K is just the quotient space T^2/G.

(i) Let Γ_0 be the set of all homeomorphisms from $\mathbb{R}^2 = \mathbb{C}$ to itself generated by tranlsations $T(z) = z + c$ for some complex number $c = m + ni$, where m and n are integers (so that $\Gamma \cong \mathbb{Z}^2$), and let Γ be the group of homeomorphisms generated by Γ_0 and the group of homeomorphisms in Γ_0 together with all maps of the form $T \circ S$, where $S(z) = \frac{1}{2} + \overline{z}$; as usual, $\overline{z}$ denotes complex conjugation. Prove that Γ_0 is a normal subgroup of Γ with index 2. [*Hint:* Prove that $S^2 \in \Gamma_0$ and that if T is a translation then STS and STS^{-1} are also translations. Why do these imply that every element of Γ can be written uniquely in the form $S^{\varepsilon} \circ T$ where T is a translation and $\varepsilon = 0$ or 1?]

(ii) If $p : \mathbb{R}^2 \to T^2$ is the usual covering space projection and $q : T^2 \to K$ is the double covering described above, show that $q \circ p$ is a covering space projection and its group of covering transformations is isomorphic to Γ. Why does this imply that $\pi_1(K)$ is nonabelian?

(iii) ($\star$) Show that Γ has no elements of finite order aside from the identity element.

Definition. If X and Y are smooth manifolds, then a covering space projection $p : X \to Y$ is said to be smooth if (1) p is smooth, (2) every point in y Y has an evenly covered open neighborhood U such that the restriction of p to each sheet is a diffeomorphism onto U.

7. Suppose that we are given a smooth connected covering space projection $p : (X, x_0) \to (Y, y_0)$ and a smooth map $f : \pi_1(W, w_0) \to (Y, y_0)$ (where W is connected) such that the image of of the map f_* on fundamental groups is contained in the image of p_*. Let $F : (W, w_0) \to (X, x_0)$ be the unique continuous lifting of f. Prove that F is smooth.

8. Suppose that we are given a smooth connected covering space projection $p : (X, x_0) \to (Y, y_0)$ and a (continuous) covering space projection $T : (X, x_0) \to (X, x_1)$. Prove that T is a diffeomorphism.