EXERCISES FOR MATHEMATICS 205C

SPRING 2011

File Number 02

DEFAULT HYPOTHESES. Unless specifically stated otherwise, all spaces are assumed to be Hausdorff and locally arcwise connected.

1. Suppose we are given groups F_1 and F_2 with subsets $X_i \subset F_i$ such that F_i is a free group on X_i for $i = 1, 2$. Using the Universal Mapping Property, prove that $F_1 * F_2$ is free on the disjoint union $X_1 \amalg X_2$.

2. If G is a group, then the *commutator subgroup* or *derived subgroup* G' (also written $[G, G]$) is the normal subgroup normally generated by all commutators; in other words, all products of the form $xyx^{-1}y^{-1}$ where $x, y \in G$. Let $p : G \to G/G'$ be the quotient projection.

(a) Prove that G/G' is abelian, and if $f : G \to A$ is a homomorphism into an abelian group A, then there is a unique homomorphism $\overline{f} : G/G' \to A$ such that $f = \overline{f} \circ p$ (this is a Universal Mapping Property for homomorphisms into abelian groups).

(b) Prove that if $\varphi : G \to K$ is a homomorphism into abelian groups which also has the Universal Mapping Property in (a), then K is isomorphic to G/G'.

(c) Suppose that the group G can be written as a free product $G_1 * G_2$. Prove that G/G' is isomorphic to $(G_1/G_1') \times (G_2/G_2')$.

3. Suppose that X is (arcwise) connected and locally simply connected, and assume further that X the union of two (arcwise) connected open subspaces U_1 and U_2 such that $U_1 \cap U_2$ is (arcwise) connected and the map $\pi_1(U_1 \cap U_2, p) \to \pi_1(X, p)$ induced by inclusion is the trivial homomorphism, where $p \in U_1 \cap U_2$. Prove that there is an isomorphism

$$(\pi_1(U_1, p)/N_1) * (\pi_2(U_2, p)/N_2) \longrightarrow \pi_1(X, p)$$

where $N_i \subset \pi_1(U_i, p)$ is the normal subgroup generated by the image of $\pi_1(U_1 \cap U_2, p) \to \pi_1(U_i, p)$. [*Hint:* Use the Universal Mapping Property.]

4. (a) Suppose that X is a union of two closed subspaces A and B such that $A \cup B$ consists of a single point p. Also assume that p has contractible open neighborhoods U and V in A and B respectively. Prove that $\pi_1(X, p)$ is the free product of $\pi_1(A, p)$ and $\pi_1(B, p)$.

(b) Given two positive integers $m, n > 1$, construct a space X whose fundamental group is $\mathbb{Z}_m * \mathbb{Z}_n$.

5. Let $p : E \to X$ be a covering map (with the usual assumptions that all spaces be locally arcwise connected, but **not necessarily connected**), and let $f : A \to X$ be a subspace inclusion, where A and X are both connected and A is locally arcwise connected. Denote the pullback covering by $E|A$.

(i) Show that A is evenly covered if the induced map of fundamental groups f_* is the trivial homomorphism.

(ii) Show that if the induced map of fundamental groups f_* is onto, then $E|A$ is connected if E is connected.

(iii) Suppose that E is simply connected. Show that if the induced map of fundamental groups f_* is 1–1, then the components of $E|A$ are all simply connected.

6. Determine the number of equivalence classes of based 2-sheeted covering spaces of the Figure Eight space $S^1 \vee S^1$, and determine the number of equivalence classes of regular based 4-sheeted coverings of the same space. [*Hints:* Every subgroup of index 2 is a normal subgroup, and normal subgroups of index n are the kernels of surjective homomorphisms onto groups of order n. Up to isomorphism there are only two groups of order 4.]

7. (a) Suppose that X is the Utilities Graph with six vertices A, B, C, G, W, E and nine edges, joining each of A, B, C to each of G, W, E. Compute the fundamental group of X and find a maximal tree in X.

(b) Suppose that X_n is the complete graph on $n \geq 4$ vertices $v_1, \cdots, v_n$, with edges joining each pair of points $v_i \neq v_j$. Compute the fundamental group of X_n and find a maximal tree in X_n.

8. Let F_2 denote the free group on the generators x and y. Prove that there is a chain of subgroups

$$\cdots \subset H_n \subset H_{n-1} \subset \cdots \subset H_3 \subset F_2$$

such that for each $k \geq 3$ the subgroup H_k is free on k generators.

9. Suppose that Y is a connected graph whose fundamental group is free on n generators, and suppose that $p : X \to Y$ is a connected n-sheeted covering space projection. Find the unique positive integer m such that the fundamental group of X is free on m generators.

10. Let X_n be the graph in $\mathbb{R}^2$ whose vertices are the lattice points (p, q) where p and q are integers such that $0 \leq p, q \leq n$, and whose edges are the segments which join (p, q) to $(p, q + 1)$ if $q < n$ or join (p, q) to $(p+1, q)$ if $p < n$ (physically, this is a square grid with n rows and n columns whose lower right corner is the origin). Compute the fundamental group of X_n and determine the number of vertices in a maximal tree T_n. Describe an explicit maximal tree when $n = 3$ or 4.