EXERCISES FOR MATHEMATICS 205C

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DEFAULT HYPOTHESES. Unless specifically stated otherwise, all spaces are assumed to be Hausdorff and locally arcwise connected.

In the first two exercises we shall use the concept of local homology groups to develop criteria for showing that certain pairs of graphs cannot be homeomorphic. Recall that if $x \in X$, then the local homology groups of X at x are defined as $H_*(X, X - \{x\})$, and by excision these are isomorphic to the groups obtained if one replaces X by an arbitrary open neighborhood U of x in X.

1. Let $(X, \mathcal{E})$ be a connected graph, and suppose that v is a vertex of $\mathcal{E}$. The supplement of v — written $\mathbf{Supp}(v, \mathcal{E})$ — is defined to be the subcomplex of all vertices except v and all edges which do not have v as one of their vertices, and the *open star* $\mathbf{OpSt}(v)$ is defined to be the complement of this subcomplex. Geometrically this is just a finite union of half open intervals sharing a common end point.

(a) Prove that if $x \in X$ is not a vertex, then the local homology group $H_1(X, X - \{x\})$ is isomorphic to $\mathbb{Z}$. [*Hint:* Let E_x be the unique edge containing x, and let $\mathcal{O}_x$ be obtained from E_x by deleting its enspoints. Then $\mathcal{O}_x$ is open because its complement is the finite union of all vertices and all edges except E_x; also, $\mathcal{O}_x$ is homeomorphic to an open interval and $x \in \mathcal{O}_x$. By excision the local homology group given above is isomorphic to $H_1(\mathcal{O}_x, \mathcal{O}_x - \{x\})$; note that the deleted neighborhood $\mathcal{O}_x - \{x\}$ is homeomorphic to a disjoint union of two open intervals.]

(b) If v is a vertex of $\mathcal{E}$, define the **branching number** $B(v, \mathcal{E})$ to be the number of edges which have v as one of their vertices. Prove that $H_1(X, X - \{v\})$ is a free abelian group on $B(v, \mathcal{E}) - 1$ generators. [*Hint:* As noted above, by excision the local homology group is isomorphic to $H_1(\mathbf{OpSt}(x), \mathbf{OpSt}(x) - \{x\})$; note that $\mathbf{OpSt}(x) - \{x\}$ is homeomorphic to a disjoint union of $B(v, \mathcal{E})$ open intervals.]

Notational convention. If $x \in X$ is not a vertex, then we shall say that the branching number $B(x, \mathcal{E})$ is equal to 2. With this convention, the conclusion of (b) extends to all points in X.

(c) If $k \neq 2$ is a positive integer, explain why the number $n_k(\mathcal{E})$ of points $x \in X$ with $B(x, \mathcal{E}) = k$ is finite, and that if $(Y, \mathcal{E}')$ is another such that X and Y are homeomorphic then $n_k(\mathcal{E}) = n_k(\mathcal{E}')$. [*Hint:* If $f : X \to Y$ is a homeomorphism such that $f(x) = y$, then we have $H_*(X, X - \{x\}) \cong H_*(Y, Y - \{y\})$.]

Another Notational convention. If $(X, \mathcal{E})$ is a graph then by (c) we can define $n_k(X) = n_k(\mathcal{E})$ because this number does not depend upon the particular graph structure $\mathcal{E}$; this number is finite if $k \neq 2$ and infinite if $k = 2$. Similarly, if $k > 0$ define $V_k(X)$ to be the set of all points with branching number k. Finally, we may also define $B(x, X) = B(x, \mathcal{E})$ because the latter does not depend upon the choice of $\mathcal{E}$.

2. (*a*) Suppose that $(X_0, \mathcal{E}_0)$ and $(X_1, \mathcal{E}_1)$ are graphs, and let $f : X_0 \to X_1$ be a homeomorphism. Prove that for all positive integers n the map h sends $V_n(X_0)$ to $V_n(X_1)$. In particular, show that $V_2(X_0)$ and $V_2(X_1)$ have the same (finite) numbers of components.

(*b*) Using the notion of n-fold branch points, show that there are at least **7** homeomorphism types represented by the standard hexadecimal digits as written below (in sans-serif type):

$$\mathsf{0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad 9 \quad A \quad B \quad C \quad D \quad E \quad F}$$

Are new homeomorphism types added if we consider the remaining letters of the alphabet? Explain. — Obviously, one can formulate similar questions for a more or less arbitrary set of printed characters.

(*c*) The Figure 8 and Figure Theta spaces, corresponding to $\mathsf{8}$ and θ respectively, turn out to have the same homotopy type (see the comments below), but neither is a deformation retract of the other, and in fact neither is homeomorphic to a subspace of the other. Prove the last assertion in the preceding sentence. [*Hint:* Suppose more generally that we have 1-dimensional graphs A and X such that A is homeomorphic to a subset of X, and let $x \in A$. Modify arguments from the previous exercises to show that $B(x, A) \leq B(x, X)$, and explain why this shows that the Figure Eight cannot be a subset of the Figure Theta and vice versa by describing the sets V_n(Figure Eight) and V_n(Figure Theta) for $n > 2$.]

Note. In fact, each of these spaces is a deformation retract of $\mathbb{R}^2 - \{\text{two points}\}$. A proof of this for the Figure Eight is described on page 462 of Munkres, and as noted in Example 3 on that page one can give a similar argument for the theta space. See also the discussion and drawings on pages 132–133 of Lee, *Introduction to Topological Manifolds*.

3. Suppose that a space X is the union of two open arcwise connected subsets U and V and the intersection $U \cap V$ is nonempty but **not** arcwise connected. Choose a base point $p \in U \cap V$. Prove that both $H_1(X) \cong H_1(X, \{p\})$ and $\pi_1(X, p)$ have infinite order, and hence the conclusion of the Seifert-van Kampen Theorem fails very badly and systematically if one drops the assumption that the intersection be arcwise connected.

4. ($\star$) Suppose that $X \subset S^2$ is a union of two simple closed curves C_1 and C_2 (homeomorphic to S^1) such that their intersection is a single point (hence X is homeomorphic to a Figure Eight). Prove that $S^2 - X$ has three components U, V, W such that the boundary of U is C_1, the boundary of V is C_2, and the boundary of W is X.

5. (*a*) If $X \subset S^2$ is homeomorphic to a tree, prove that the reduced homology groups of $\widetilde{H}_*(S^2 - X)$ are trivial. [*Hint:* We know this if X has a single edge. Proceed by induction and use the fact that $X = X_0 \cup E^*$, where E^* is an edge such that exactly one vertex lies in X_0.]

(*b*) Suppose that X is a connected graph whose fundamental group is a free group on m generators, and suppose that $A \subset S^2$ is homeomorphic to X. Prove that $S^2 - A$ has $m + 1$ connected components.

6. Suppose that $U \subset \mathbb{R}^m$ is a nonempty open subset and $m > n$. Prove that there is no continuous 1–1 mapping from U into $\mathbb{R}^n$. [*Hint:* Explain why it suffices to consider the case where $U = V \times V'$ where V and V' are open in $\mathbb{R}^n$ and $\mathbb{R}^{m-n}$ respectively and each contains $\mathbf{0}$. Let v be a unit vector in $\mathbb{R}^{m-n}$, and show that there is some $\delta > 0$ such that if $t \in (-\delta, \delta) - \{0\}$ then every point of the form tv also lies in the image of $V \times \{\mathbf{0}\}$. What do we know about the image of the latter subset?]

7. The Phragmén-Brouwer Property is the following statement.

> *Let X be a locally arcwise connected space, and let A and B be two nonempty proper closed subsets such that $A \cap B = \emptyset$. If $p, q \in X - (A \cup B)$ are such that p and q lie in the same arc components of $X - A$ and $X - B$, then p and q lie in the same arc component of $X - (A \cup B)$.*

(a) Prove that if X is arcwise connected and $H_1(X) = 0$, then X satisfies the Phragmén-Brouwer Property. [*Hint:* Why do we have $X = (X - A) \cup (X - B)$? Note that $X - (A \cup B) = (X - A) \cap (X - B)$.]

(b) Prove that T^2 does not have the Phragmén-Brouwer Property. [*Hint:* Let A and B be the circles $\{\pm 1\} \times S^1$ and try to find candidates for p and q.]

8. (a) Suppose that X and X' have regular cell complex structures $\mathcal{E}$ and $\mathcal{E}'$ respectively. Explain why the product $X \times X'$ has a regular cell complex structure whose k-cells have the form $A \times B$, where A is a p-cell in $\mathcal{E}$ and B is a $(k-p)$-cell in $\mathcal{E}'$.

(b) In the setting of (a), prove that the Euler characteristic of $X \times X'$ is given by $\chi(X) \cdot \chi(X')$. [*Hint:* Use (a) to derive a formula for the number of k-cells in the product, expressed in terms of the numbers of p-cells in $\mathcal{E}$ and q-cells in $\mathcal{E}'$, where $k = p + q$.]

(c) Let X be a connected graph such that $H_1(X)$ is free abelian on more than one generator. Using Euler characteristics, prove that $H_2(X \times X)$ must be nontrivial. [*Hint:* Since X is a retract of $X \times X$, it follows that $H_1(X \times X)$ contains a free abelian group on at least two generators. We also know that $X \times X$ has a 2-dimensional cell complex structure, so it has no homology in dimensions greater than 2. Why is the Euler characteristic of $X \times X$ positive?]

9. If $f : S^n \to S^n$ is a continuous map where $n \geq 1$, then the isomorphism $H_n(S^n) \cong \mathbb{Z}$ implies that the homology map $f_* : H_n(S^n) \to H_n(S^n)$ is multiplication by some uniquely determined integer d, and this integer is called the **degree** of f.

(a) Given f as above, prove that if $d \neq 0$ then f is onto. [*Hint:* If $p \in S^n$ is not in the image of f, what can we say about $S^n - \{p\}$)?]

(b) If $n = 1$ and $f(z) = z^d$ (algebraic exponentiation), explain why the degree of f is equal to d. [*Hint:* Use the natural map relating π_1 to H_1.]

(c) If f and g are both continuous maps from S^n to itself and their respective degrees are a and b, show that the degree of $f \circ g$ is equal to ab.

(d) Let σ be the homomorphism from the suspension $\Sigma(S^n)$ to S^{n+1} which maps the equivalence class $[x, t]$ to the point $\left(\sqrt{1-t^2}\, x, t\right) \in S^{n+1} \subset \mathbb{R}^{n+1} \times \mathbb{R}$, so that σ takes the upper and lower cones in the suspension to the upper and lower hemispheres of the sphere. Given a continuous map $f : S^n \to S^n$, define $S(f)$ to be the composite $\sigma \circ \Sigma(f) \circ \sigma^{-1}$. Using the suspension isomorphism from $H_{n+1}\left(\Sigma(S^n)\right)$ to $H_n(S^n)$, profe that $S(f)$ and f have the same degree. Why does this imply that for every integer d and $n \geq 1$, there is a continuous map from S^n to itself with degree d? [*Hint:* Use the suspension isomorphism to prove that $\Sigma(f)$ induces multiplication by d on $H_{n+1}\left(\Sigma(S^n)\right)$.]

(d) If f is a continuous map from S^n to itself and h is a homeomorphism from S^n to itself, prove that the degrees of f and $h \circ \circ h^{-1}$ are equal.

(e) Given an $(n+1)\times(n+1)$ orthogonal matrix A, let f_A be the induced self-map of S^n, and let $A \oplus 1$ be the $(n+2)\times(n+2)$ matrix given in block form by

$$\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} .$$

Prove that $f_{A\oplus 1} = S(f_A)$ and hence that the degrees of f_A and $f_{A\oplus 1}$ are equal.

(g) In the preceding part of this problem, assume that A is a diagonal matrix (whose diagonal entries are necessarily equal to ± 1). Prove that the degree of f_A is the number of negative diagonal entries, and that if $A = -I$ then the degree of f_A is equal to $(-1)^{n+1}$.

(h) It is known that if A is an orthogonal matrix with determinant equal to $+1$, then there is a continuous curve $A(t)$ such that $A(0) = I$, $A(1) = A$, and each $A(t)$ is an orthogonal matrix. One proof of this is described in the next paragraph. Using this fact, prove that (1) if $\det A = +1$ then the degree of f_A is also $+1$, and also (2) if A is an arbitrary orthogonal matrix, then $\det A$ is equal to the degree of f_A.

FOOTNOTE. Here is one way to prove the assertion about orthogonal matrices of determinant $+1$ in (h) based on a standard normal form for orthogonal matrices. As noted in Appendix D of the online document

`http://math.ucr.edu/≃res/math205A/gentopnotes2008.pdf`

one can always find an orthogonal matrix P such that $P^{-1}AP$ is a block sum of 2×2 rotation matrices and 1×1 matrices whose unique entries are ± 1. Since the negative of the 2×2 identity matrix is also rotation through π radians, we may as well assume that the block sum has at most one 1×1 matrix with an entry of -1, so if $\det A = 1$ then we can assume that the block sum consists of 2×2 rotation matrices and possibly 1×1 identity matrices. If $D(\theta_i)$ is a block sum given by rotation through θ_i, then clearly one has a continuous curve in the space of orthogonal matrices joining the 2×2 identity matrix to $D(\theta_i)$ — namely, $\gamma_i(t) = D(t\theta_i)$. The block sum of the curves $\gamma_i(t)$ defines a curve $C(t)$ in the space of orthogonal matrices joining the identity to $P^{-1}AP$, and $PC(t)P^{-1}$ will then be a curve in the space of orthogonal matrices joining the identity matrix to A.■

10. (a) Suppose we are given a product of spheres

$$S^{n_1} \times \cdots \times S^{n_k}$$

where we arrange the factors for the sake of convenience so that $1 \le n_1 \le \cdots \le n_k$. Explain why

$$\bigoplus_{q=0}^{\infty} H_q(S^{n_1} \times \cdots \times S^{n_k})$$

is a free abelian group of rank 2^k. [*Hint:* Use Exercises 04.4 and induction on the number of factors.]

(b) Suppose we have homeomorphic products of spheres

$$S^{n_1} \times \cdots \times S^{n_p} \qquad \text{and} \qquad S^{m_1} \times \cdots \times S^{m_q}$$

where $1 \le n_1 \le \cdots \le n_p$ and $1 \le m_1 \le \cdots \le m_q$. Prove that $p = q$ and $m_i = n_i$ for all i. [*Hint:* It is useful to prove a generalization to products of spheres with isomorphic homology groups in all dimensions. Use (a) to prove that $p = q$, and note that if the first nonzero positive dimensional homology arises in dimension r then $n_1 = r = m_1$. Why does this imply that $S^{n_2} \times \cdots \times S^{n_p}$ and $S^{m_2} \times \cdots \times S^{m_q}$ have isomorphic homology groups? Use this to construct an inductive argument on the number of factors.]

11. Suppose that X and Y are connected spaces with $p \in X$ and $q \in Y$. Prove that $H_1(X \times Y) \cong H_1(X) \oplus H_1(Y)$. [*Hint:* What do we know about the fundamental group of $\big(X \times Y, (x, y)\big)$ and what does this imply about $H_1(X \times Y)$?]

12. Suppose that $(X, \mathcal{E})$ is a connected regular cell complex, and suppose that $Y \to X$ is an n-sheeted covering space projection (where n is finite).

(a) Prove that there is a regular cell complex structure $\mathcal{E}'$ on Y such that each cell in $\mathcal{E}'$ projects homeomorphically to a cell in X. [*Hint:* Look at the restriction of the covering to a cell in X. Why is the cell evenly covered?]

(b) Using this cell structure, show that the Euler characteristics of X and Y are related by $\chi(Y) = n \cdot \chi(X)$.

(c) It is known that $\mathbb{RP}^2$ is homeomorphic to a 2-dimensional polyhedron. Using this fact, prove that its Euler characteristic is equal to 1 and that $H_2(\mathbb{RP}^2)$ has no infinite cyclic summands. [*Hint:* Explain why the identity $\pi_1(\mathbb{RP}^2) \cong \mathbb{Z}_2$ implies that $H_1(\mathbb{RP}^2) \cong \mathbb{Z}_2$.]

(d) $(\star)$ Using the methods on pages 6–7 of `cell-euler.pdf`, show that if X has an n-dimensional cell complex structure, then $H_n(X)$ is torsion free; in other words, it has no nonzero elements of finite order. [*Hint:* Work inductively. The conclusion is true if X is $(n-1)$ dimensional, or equivalently an n-dimensional complex with no n-cells. Assume it is true for complexes with at most k cells of dimension n, let X be a complex with $k+1$ cells of dimension n, and let X_0 be obtained by deleting one of the n-dimensional cells in X. Use the fact that $H_*(X, X_0) \cong H_*(D^n, S^{n-1})$.]

NOTE. This is relevant to (c) for it implies that $H_2(\mathbb{RP}^2)$ is torsion free, and since it has no infinite cyclic summands it must be equal to zero.