Free groups and free products

(Topic 5 background).

Def. X a set $i : X \longrightarrow F$ 1-1, where F is a group. Then we say that F is freely generated by $i[X]$ if

① $i[X]$ generates F

② given a map of sets $j : X \longrightarrow G$ where G is a group, there is a unique homomorphism $h : F \longrightarrow G$ s.t. $j = h \circ i$.

Consequence 1 $i : X \to F$ & $i' : X \longrightarrow F'$ determine free groups $\Rightarrow$ there is a unique isomorphism $\alpha : F \longrightarrow F'$ s.t. $\alpha \circ i = i'$.

Consequence 2 $i : X \to F$ & $j : Y \to F$ determine free groups $\Rightarrow \#(X) = \#(Y)$.

<u>Proof of 1</u> The only homomorphism $h : F \to F$ s.t. $h \circ i = i$ is id_F.

By ②, we have unique homs $\alpha : F \to F'$, $\beta : F' \to F$ s.t. $i' = \alpha i$ and $i = \beta i'$.

Hence

$$i = \beta i' = \beta \alpha i = 1 i \Rightarrow \beta\alpha = id_F$$

$$i' = \alpha i = \alpha \beta i' = 1 i' \Rightarrow \alpha\beta = id_{F'}.$$

Hence α & β are isomorphisms.

<u>Proof of 2</u>

There are $2^{\#(X)}$ homs $F \to \mathbb{Z}_2$ determined by their values on $i[X]$.

Hence X, Y finite $\Rightarrow 2^{\#(X)} = 2^{\#(Y)} \Rightarrow \#(X) = \#(Y)$. Say one of X, Y is finite and the other is infinite. Then we would have $\underbrace{2^{\#(X)}}_{\text{finite}} = \underbrace{2^{\#(Y)}}_{\text{infinite}}$ a contradiction.

Finally, suppose X, Y both infinite. Then every elt of F has the form

$$x_1^{\varepsilon_1} \cdots x_k^{\varepsilon_k} \qquad \varepsilon_j = \pm 1$$

where $x_j \in X$. Likewise every elt. has the form $y_1^{\delta_1} \cdots y_k^{\delta_k}$. The number of such sequences is $\#(X)$ and $\#(Y)$ respectively (why?). So if F is free on X and X is infinite, then $\#(F) = \#(X)$. Hence $\#(X) = \#(Y)$ if X and Y are both infinite. ■

One can construct the free group explicitly, and it turns out that its nontrivial elements are given by reduced words $x_1^{\varepsilon_1} x_2^{\varepsilon_2} \cdots x_k^{\varepsilon_k}$

such that if $x_i = x_{i-1}$ then $\varepsilon_i \neq -\varepsilon_{i-1}$ ($\Rightarrow$ no obvious cancellations)

Free products.

G_1 + G_2 groups. $\varphi_i : G_i \longrightarrow S$.
homs.

S is a free product if

① $\varphi_1[G_1] \cup \varphi_2[G_2]$ generates S

② If H is a group and $h_i: G_i \longrightarrow H$ are homs, then there is a unique hom $f: S \longrightarrow H$ s.t. $h_i = f \circ \varphi_i$.

Similar consequences

⓪ φ_i are 1-1 [take h_i = identity on G_i, h_j = constant if $j \neq i$; then $f \circ \varphi_i = id$ $\Rightarrow \varphi_i$ is 1-1].

① $(\varphi_1, \varphi_2; S)$ unique up to isomorphism.

② The elements of $G_1 * G_2$ are describable as reduced words in G_1 and G_2. (See Topic 5)