Homology of connected graphs

<u>Notation</u> $(X, \mathcal{E})$ is a connected graph and we have a linear ordering of its vertices.

<u>CLAIM 1</u>. $C_0(X,\mathcal{E})/\mathrm{Im}\, d_1 \cong \mathbb{Z}$, and if v is a vertex then v maps to a generator.

<u>Proof</u> <u>Step 1.1</u> Define an <u>augmentation</u> map $\varepsilon : C_0(X,\mathcal{E}) \longrightarrow \mathbb{Z}$ sending each vertex to 1 ~~a generator~~. We claim first that ~~[illegible]~~, so $\varepsilon \mid \mathrm{Image}\, d_1 = 0$, so that we get an onto map $\bar{\varepsilon} : C_0(X,\mathcal{E})/\mathrm{Image}\, d_1 \to \mathbb{Z}$ sending each vertex to 1. To show $\varepsilon \mid \mathrm{Image}\, d_1 = 0$ it suffices to prove $\varepsilon \circ d_1$ is zero on the free generators for $C_1(X,\mathcal{E})$. Let E be an edge with endpoints v_i, v_j where $i<j$. Then $\varepsilon d_1(E) = \varepsilon(v_j - v_i) = \varepsilon(v_j) - \varepsilon(v_i) = 0$.

Step 1.2 Show that if $v \neq v'$ are vertices, then $v' - v \in \text{Image } d_1$. Take an endpoint edge-path curve $F_1, \ldots, F_m$ joining v to v', so the endpoints of F_i are w_i and w_{i-1} with $v' = w_m$ and $v = w_0$.

Form the chain $\sum \varepsilon_i F_i$ where $\varepsilon_i = \pm 1$ is chosen to be $\begin{cases} +1 & \text{if } w_{i-1} \text{ precedes } w_i \text{ in vtx ordering} \\ -1 & \text{otherwise.} \end{cases}$

Then $d \sum \varepsilon_i F_i = \sum w_i - w_{i-1} = v' - v$.

Step 1.3 Finish the proof by showing that if $\sum n_i v_i$ satisfies $\sum n_i = 0$, then $\sum n_i v_i \in \text{Image } d_1$. For each $i > 1$, choose a chain c_i so that $d_1 c_i = v_i - v_1$. Then

$$\sum_i n_i v_i = \sum_i n_i v_i - \sum_i n_i v_1 \quad (\text{since } \textstyle\sum n_i = 0)$$

$$= \sum_{i>1} n_i (v_i - v_1) = \sum_{i>1} n_i d_1 c_i = d_1 \Big(\sum_{i>1} n_i c_i \Big).$$

CLAIM 2. $H_1(X, \mathcal{E})$ is free abelian on $1 - \chi(X, \mathcal{E})$ generators.

Proof. Given a fin gen free abelian group A with a set of free gens, let $Vec(A)$ be the rational vector space with the same set of basis elts (= free generators over the rationals). If $T: A \to A'$ is a homomorphism of such objects, then one gets an associated linear transformation $Vec(T)$: $Vec(A) \to Vec(A')$.

① If $K = \text{Kernel}\, T$, then $Vec(K) \cong \text{Kernel}\, Vec(T)$.

② If $J = \text{Image}\, T$, then $Vec\, J \cong \text{Image}\, Vec(T)$.

(left to the reader)

Since $H_1(X, \mathcal{E}) = \text{Kernel}\, d_1$, it follows that $H_1(X, \mathcal{E})$ is free abelian on

$$\dim \text{Kernel}\, Vec(d_1) \text{ generators.}$$

By Claim 1 we know $\text{Kernel}\, \varepsilon = \text{Image}\, d_1$.

Therefore it follows that $Vec($ $\uparrow$ THIS SUBGROUP $)$

must have dimension equal to #vertices$(X)-1$.

By linear algebra,

$$\#\text{edges}(X) = \dim \operatorname{Vec} C_1(X,\mathcal{E}) =$$

$$\operatorname{rank} \operatorname{Vec}(d_1) + \operatorname{nullity} \operatorname{Vec}(d_1) =$$

$$(\#\text{vertices}(X) - 1) + \operatorname{rank} H_1(X,\mathcal{E})$$

which simplifies to

$$\operatorname{rank} H_1(X,\mathcal{E}) = 1 - \#\text{vertices}(X) + \#\text{edges}(X)$$

$$= 1 - \chi(X,\mathcal{E}).$$

In other words,

$H_1(X,\mathcal{E})$ is the abelianization of $\pi_1(X,x_0)$.