Embedding graphs in $\mathbb{R}^3$

THEOREM. Let $(X, \mathcal{E})$ be a connected graph. Then X is homeomorphic to a subset of $\mathbb{R}^3$ (in fact, the underlying space of a simplicial complex).

Key Step. 1 $\mathbb{R}^3$ has an infinite sequence of (isolated) points such that no 4 are coplanar (affinely dependent).

Sketch Obviously $\mathbb{R}^3$ contains 4 such points: $\vec{0}$ and the standard unit vectors. Suppose we have a set Y of n such points. Let P_i, $1 \leq i \leq \binom{n}{3}$, be the planes determined by triples of such points. Then $\mathbb{R}^3 - \cup P_i$ is an open dense subset. Add to Y a point $z \in \mathbb{R}^3 - \cup P_i$ such that $|z| \geq n+1$. Continue to define the desired ∞ sequence recursively.

<u>Proof of Theorem</u> Take the infinite sequence described above and map the vertizes of $\mathcal{G}$ to distinct points in the sequence.

<u>Key Step 2</u> For the sequence in Key Step 1, the different segments joining two distinct vertizes ~~are~~ can only meet at end points.

$\boxed{\text{Case A}}$ The segments have ~~a~~ one end point in common, so one has end points x, y and the other has end points y, z. When can we have $tx + (1-t)y = sy + (1-s)z$? By affine independence* this only happens if $t=0$, $1-s=0$ and $1-t=s$; i.e., if $s=1-t=1$.

* barycentric coords of x, y, z all equal in the two expressions

$\boxed{\text{Case B}}$ The segments have no end points in common. Say they join x, y and z, w.

then we have

$$t x + (1-t) y = s z + (1-s) w$$

and by affine independence* these yield

* as on the prev. page.

$$t = 0,\ (1-t) = 0,\ 0 = s,\ 0 = (1-s).$$

equate bary centric coords of

x y z w

So in fact the lines joining x, y and z, w cannot have any points in common.

Conclusion of Theorem Proof We have a 1-1 map on vertices, say φ. Given edge E in $\mathcal{E}$ with endpoints a and b, map E 1-1 continuously to the line segment joining $\varphi(a)$ to $\varphi(b)$. Key Step 2 implies this map is 1-1 continuous and hence is a homeomorphism onto its image. ∎