Axioms for a quasi-singular homology theory

This document gives a set of axioms for certain homology theories such as the singular homology theory of S. Eilenberg and N. Steenrod. As in the latters' book, *Foundations of Algebraic Topology*, one objective is to provide a framework from which one can apply the theory to a fairly wide range of classical questions and results in topology. It is possible to prove that the axioms uniquely characterize singular homology for a wide class of spaces which includes polyhedra, open subsets of $\mathbb{R}^n$ and various spaces of continuous or smooth functions with standard sorts of topologies, but doing so requires methods which go even beyond the topics in Bredon's book. For the sake of completeness, an outline of one proof (with references) appears in `homology-uniqueness.pdf`.

Comparing the axiomatic and constructive approaches

In this course, we have adopted an axiomatic approach because of time limitations and a priority for showing how homology theory applies to topological problems of independent interest. Such an approach follows one important thread in the book of Eilenberg and Steenrod, but our approach differs from theirs in some key respects.

Although we are treating singular homology theory axiomatically, this does not mean that the standard construction in textbooks is relatively unimportant. The next step in any further study of algebraic topology should be to go through the explicit construction of singular homology theory (which requires more input from the algebraic theory of chain complexes) and the proofs that that the standard description of singular homology does satisfy all the axioms given below.

It is also important to note that singular homology theory has many additional properties and extra structures beyond those described in our axioms, so an understanding of the explicit, standard construction of singular homology is indispensable for studying algebraic topology in greater depth.

Finally, we should note that there is a great deal of redundancy in our axioms. Ultimately one would like to have a set of axioms that have little or no logical interdependence, but for our purposes it seems best to start with a set of axioms which can be used fairly simply and directly to prove nontrivial results. At this point, proving that some of these axioms imply the others would cut seriously into the time available for discussing the applications we want to cover.

Pairs of topological spaces

As in our discussion of simplicial homology, it is useful (in fact necessary) to work with **pairs** consisting of an object and a subobject.

Definition. A *pair* of topological spaces is an ordered pair (X, A) where A is a subspace of X. A (continuous) map of pairs $f : (X, A) \to (Y, B)$ is given by a continuous mapping $f : X \to Y$ such that $f[A] \subset B$. — It follows immediately that pairs of spaces and maps of pairs form a category.

We can embed the category of topological spaces and continuous mappings into the category of pairs by sending X to $(X, \emptyset)$, and often we shall use X to denote this pair. Given two pairs (X, A) and (Y, B), their product in the category of pairs is given by $(X \times Y,\ X \times B \cup A \times Y)$. With this definition, the cartesian product of two maps of pairs becomes a map of pairs (verify this!). In particular, if $B = \emptyset$ then we can write $(X, A) \times Y = (X \times Y, A \times Y)$.

There is a natural embedding of the category of spaces with base points into the category of pairs of spaces. Specifically, if (X, x_0) is a space with base point we take the corresponding pair $(X, \{x_0\})$, and similarly base point preserving maps define maps of the associated pairs of spaces.

Another important construction in the category of pairs is the natural map $j_{(X,A)} : (X, \emptyset) \to (X, A)$ which is the identity on X (and the empty function on $\emptyset$).

The data for an abstract singular homology theory

1. For each pair of spaces (X, A) and each integer q we are given an abelian group $H_q(X, A)$ and a homomorphism $\partial_q : H_q(X, A) \to H_{q-1}(A)$.

2. For each decomposition of a space $X = U \cup V$ such that $\text{Interior}(U) \cup \text{Interior}(V) = X$ we are given a homomorphism $\Delta : H_q(X) \to H_{q-1}(U \cap V)$.

Note that the condition on U and V is automatically satisfied if U and V are both open in X, and in fact this is probably the most important special case for our purposes.

3. For each map of pairs $f : (X, A) \to (Y, B)$ and each integer a we are given a homomorphism $f_* : H_q(X, A) \to H_q(Y, B)$.

Sometimes such data are called a ∂-functor (pronounced "dell functor") from the category of pairs of topological spaces and maps of pairs to the category of abelian groups. The maps ∂ are frequently called *connecting homomorphisms* or *switchback homomorphisms*.

It is also useful (but actually redundant) to assume a relationship between the fundamental group and the first homology group which is a formal version of the relationship between the fundamental group and first homology group of a connected graph, and for this we need the following:

4. There is a family of group homomorphisms $h(X, x) : \pi_1(X, x) \to H_1(X, \{x\})$.

Finally, we want some sort of relationship between our axiomatized homology theory and the simplicial homology groups we have been considering.

5. If $(P, \mathbf{K})$ is a simplicial complex (strictly speaking, with an ordering of the vertices) and $(Q, \mathbf{L})$ is a subcomplex of $(P, \mathbf{K})$, there is a sequence of homomorphisms $\theta_{(\mathbf{K},\mathbf{L})} : H_q(\mathbf{K}, \mathbf{L}) \to H_q(P, Q)$.

Functoriality and naturality

One reason the fundamental group is so useful is that it is functorial with respect to continuous maps; if f and g are two composable maps of pointed spaces and f_* and g_* then $(g \circ f)_* = g_* \circ f_*$ and if f is an identity map then so is f_*. We want a similar sort of condition for homology groups.

(A.1) *If $f : (X, A) \to (Y, B)$ and $g : (Y, B) \to (Z, C)$ are maps of pairs, then $(g \circ f)_* = g_* \circ f_*$. If f is the identity map on (X, A), then f_* is the identity on $H_q(X, A)$ for all integers q.*

One important consequence of this is that if f is a homeomorphism of pairs, then the homology maps f_* are isomorphisms (if $g = f^{-1}$, then we have $g_* = (f_*)^{-1}$ by the same sort of argument which proves an analogous result for fundamental groups).

We have only defined maps in simplicial homology groups from $H_*(\mathbf{K}, \mathbf{L})$ to $H_*(\mathbf{K}', \mathbf{L}')$ when $\mathbf{K}$ and $\mathbf{L}$ are subcomplexes of $\mathbf{K}'$ and $\mathbf{L}'$ respectively, but we would like θ to be natural with respect to such maps of pairs.

(A.2) *If we are given inclusions as in the preceding paragraph and the underlying spaces are given by (P, Q) and (P', Q') respectively, then for each integer q the diagram*

$$\begin{array}{ccc} H_q(\mathbf{K}, \mathbf{L}) & \longrightarrow & H_q(\mathbf{K}', \mathbf{L}') \\ \big\downarrow \theta & & \big\downarrow \theta \\ H_q(P, Q) & \xrightarrow{\partial} & H_q(P', Q') \end{array}$$

is commutative, where the horizontal arrows come from subcomplex or subspace inclusions of pairs.

We also want the homology homomorphisms induced by a map f to be compatible with the maps ∂ in the following sense:

(A.3) *If $f : (X, A) \to (Y, B)$ is a map of pairs, then for each integer q the diagram*

$$\begin{array}{ccc} H_q(X, A) & \xrightarrow{\partial} & H_{q-1}(A) \\ \big\downarrow f_* & & \big\downarrow f_* \\ H_q(Y, B) & \xrightarrow{\partial} & H_{q-1}(B) \end{array}$$

is commutative.

We also want a similar property for the map $\Delta : H_{q+1}(U \cup V) \to H_q(U \cap V)$ given in the data. It turns out that this is a consequence of other axioms, but we shall not try to do so.

(A.4) *If we are given spaces $X_i = U_i \cup V_i$ for $i = 1, 2$, where* $\mathrm{Interior}(U_i) \cup \mathrm{Interior}(V_i) = X_i$, *and $f : X_1 \to X_2$ is a continuous map which maps U_1 and V_1 into U_2 and V_2 respectively, then for all integers q the diagram*

$$\begin{array}{ccc} H_q(X_1) & \xrightarrow{\Delta} & H_{q-1}(U_1 \cap V_1) \\ \big\downarrow f_* & & \big\downarrow f_* \\ H_q(X_2) & \xrightarrow{\Delta} & H_{q-1}(U_2 \cap V_2) \end{array}$$

is commutative.

We also want the connecting homomorphisms from singular and simplicial homology to be compatible:

(A.5) *If $(Q, \mathbf{L})$ is a subcomplex of the simplicial complex $(P, \mathbf{K})$, the for all integers q the diagram*

$$\begin{array}{ccc} H_q(\mathbf{K}, \mathbf{L}) & \xrightarrow{\partial} & H_{q-1}(\mathbf{L}) \\ \big\downarrow \theta & & \big\downarrow \theta \\ H_q(P, Q) & \xrightarrow{\partial} & H_{q-1}(Q) \end{array}$$

is commutative.

Finally, we want a naturality property of the map h fundamental groups to 1-dimensional homology.

(A.6) *If $f : (X, x) \to (Y, y)$ is a continuous base point preserving map of arcwise connected spaces, then the diagram*

$$\begin{array}{ccc} \pi_1(X,x) & \xrightarrow{f_*} & \pi_1(Y,y) \\ \downarrow h & & \downarrow h \\ H_1(X,\{x\}) & \xrightarrow{f_*} & H_1(Y,\{y\}) \end{array}$$

is commutative.

Usually a mapping like h is called a Hurewicz (hoo-RAY-vich) homomorphism.

Exactness

In simplicial homology we have a long exact sequence associated to a pair $(Q, \mathbf{L}) \subset (P, \mathbf{K})$, and we want a similar exact sequence in singular homology for the pair (P, Q). In fact, we want such a sequence for an arbitrary pair, and we want it to have good compatibility properties. We shall start with existence:

(B.1) *If (X, A) is a pair of topological spaces then there is a long exact sequence*

$$\cdots \quad H_{q+1}(X,A) \xrightarrow{\partial} H_q(A) \xrightarrow{i_*} H_q(X) \xrightarrow{j_*} H_q(X,A) \xrightarrow{\partial} H_{q-1}(A) \quad \cdots$$

which extends indefinitely to the left and to the right for all integers q. In this sequence i_* is induced by the inclusion map $A \to X$, and j_* is induced by the inclusion of pairs from X to (X, A).

We actually need two types of compatibility; namely, compatibility with respect to continuous maps of pairs and compatibility with the maps θ passing from simplicial to singular homology. These will be stated individually, and they all follow from (B.1), the first group of axioms, and the known properties of simplicial homology groups (hence they are redundant).

(B.2) *If we are given a continuous map of pairs $f : (X, A) \to (Y, B)$, then we have the following commutative ladder diagram in which the two rows are exact:*

$$\begin{array}{ccccccccccc} \cdots & H_{q+1}(X,A) & \xrightarrow{\partial} & H_q(A) & \xrightarrow{i_*} & H_q(X) & \xrightarrow{j_*} & H_q(X,A) & \xrightarrow{\partial} & H_{q-1}(A) & \cdots \\ \cdots & \downarrow f_* & & \downarrow f_* & & \downarrow f_* & & \downarrow f_* & & \downarrow f_* & \\ \cdots & H_{q+1}(Y,B) & \xrightarrow{\partial} & H_q(B) & \xrightarrow{i_*} & H_q(Y) & \xrightarrow{j_*} & H_q(Y,B) & \xrightarrow{\partial} & H_{q-1}(B) & \cdots \end{array}$$

This statement turns out to be a fairly straightforward consequence of (A.1) and (B.1).

(B.3) *Let $(X, \mathbf{K})$ be a simplicial complex, and let $(A, \mathbf{L})$ be a subcomplex. Then there is a commutative ladder as below in which the horizontal lines represent the long exact* *homology sequences of pairs and the vertical maps are the natural transformations from simplicial to singular homology.*

$$\begin{array}{ccccccccccccc}
\cdots & H_{q+1}(\mathbf{K},\mathbf{L}) & \xrightarrow{\partial} & H_q(\mathbf{L}) & \xrightarrow{i_*} & H_q(\mathbf{K}) & \xrightarrow{j_*} & H_q(\mathbf{K},\mathbf{L}) & \xrightarrow{\partial} & H_{q-1}(\mathbf{L}) & \cdots \\
\cdots & \Big\downarrow \theta & & \Big\downarrow \theta & & \Big\downarrow \theta & & \Big\downarrow \theta & & \Big\downarrow \theta & \\
\cdots & H_{q+1}(X,A) & \xrightarrow{\partial} & H_q(A) & \xrightarrow{i_*} & H_q(X) & \xrightarrow{j_*} & H_q(X,A) & \xrightarrow{\partial} & H_{q-1}(A) & \cdots
\end{array}$$

This statement turns out to be a fairly straightforward consequence of (A.2), (B.1) and the long exact simplicial homology sequence for the pair $(\mathbf{K},\mathbf{L})$.

There will eventually be one more axiom concerning long exact sequences (the Mayer-Vietoris Sequence Axiom), but we we postpone its statement because if fits more naturally into another group of assumptions.

Homotopy invariance and compact support

We have seen that the simplicial homology of an n-simplex is isomorphic to the simplicial homology of a one point complex, with $H_0(P,\mathbf{K}) \cong \mathbb{Z}$ and $H_q(P,\mathbf{K}) = 0$ for $q \neq 0$, and our proof used an algebraic analog of the standard topological homotopy contracting the simplex into a vertex. Also, for spaces with base points we know that base point preservingly homotopic maps induce the same homomorphisms of fundamental groups. The Homotopy Invariance Axiom is a very strong analog of these phenomena. For the sake of completeness, we note that a two maps of pairs $f_0, f_1 : (X,A) \to (Y,B)$ are homotopic as maps of pairs if there is a continuous map

$$H : (X,A) \times [0,1] \;=\; \big(X \times [0,1],\, A \times [0,1] \big) \;\longrightarrow\; (Y,B)$$

such that the restriction of H to $X \times \{i\}$ is f_i for $i = 0, 1$.

(C.1) *If $f_0, f_1 : (X,A) \to (Y,B)$ are homotopic as maps of pairs, then their induced homology homomorphisms $f_{0\,*}$ and $f_{1\,*}$ are equal.*

One important consequence of this is that if f is a homotopy equivalence of pairs in the appropriate sense, then the homology maps f_* are isomorphisms.

Two other important properties of the fundamental group are that each class in $\pi_1(X,x)$ comes from $\pi_1(C,x)$ for some compact subset C containing x, and if C has these properties then a class $\alpha \in \pi_1(C,x)$ maps to the trivial element of $\pi_1(X,x)$ (under the inclusion induced map) if and only if there is some compact subset D such that $C \subset D \subset X$ and α maps to the trivial element of $\pi_1(D,x)$ (under the inclusion induced map). We want a similar property for singular homology which is sometimes called a *compact supports property*:

(C.2) *If $u \in H_q(X,A)$ then there is a pair of compact subsets (C,C') such that u is in the image of the map from $H_q(C,C')$ to $H_q(X,A)$ induced by inclusion. Furthermore, if (C,C') is a pair of compact subsets and $v \in H_q(C,C')$ maps trivially to $H_q(X,A)$ under the map induced by inclusion, then there is some compact pair (D,D') such that $C \subset D$, $C' \subset D'$, and v maps trivially to $H_q(D,D')$ under the map induced by inclusion.*

Although this does not correspond to any of the original axioms due to Eilenberg and Steenrod, it plays a very important role in the applications of singular homology theory and the proof of its uniqueness.

Normalization properties

We want singular homology to coincide with simplicial homology for polyhedra/simplicial complexes (with ordered vertices), and we also want to describe H_0, H_1 and negative-dimensional homology groups in terms of objects we already understand.

(D.1) *If $(Q, \mathbf{L})$ is a subcomplex of $(P, \mathbf{K})$, then for all integers q the map $\theta : H_q(\mathbf{K}, \mathbf{L}) \to H_q(P, Q)$ is an isomorphism.*

If $(P, \mathbf{K})$ is a polyhedron then we have seen that $H_0(\mathbf{K})$ is free abelian on the set of components (equivalently, arc components), and $H_*(\mathbf{K})$ splits into a direct sum of the homology groups for the various components. We want similar results for singular homology:

(D.2) *If X is written as a union of its (pairwise disjoint) arc components X_α, then the inclusion maps $i_\alpha : X_\alpha \to X$ define an isomorphism from the (weak) direct sum $\oplus_\alpha\, H_*(X_\alpha)$ to $H_*(X)$.*

The weak direct sum of the abelian groups G_α consists of all elements of $\prod_\alpha\, G_\alpha$ such that only finitely many coordinates are nonzero (with addition defined coordinatewise).

(D.3) *If X is arcwise connected, then $H_0(X) \cong \mathbb{Z}$.*

The preceding two axioms imply that if X is arbitrary, then $H_0(X)$ is isomorphic to a free abelian group on the arc components of X.

The next axiom is actually redundant, but we shall assume it for the sake of convenience:

(D.4) *If $q < 0$ then for all pairs (X, A) we have $H_q(X, A) = 0$.*

The final axiom in this group is also redundant, but it provides an important relationship between 1-dimensional homology and the fundamental group. An early version of this result was due to H. Poincaré.

(D.5) *If X is an arcwise connected space and $x \in X$, then the homomorphism $h(X, x) : \pi_1(X, x) \to H_1(X, \{x\})$ is onto and its kernel is the commutator subgroup.*

Excision

These axioms are variants of the simplicial homology isomorphism

$$H_*(\mathbf{K}_1, \mathbf{K}_1 \cap \mathbf{K}_2) \;\longrightarrow\; H_*(\mathbf{K}_1 \cup \mathbf{K}_2, \mathbf{K}_1)$$

we discussed previously. Each of the three versions below is basically equivalent to the others.

(E.1) *Suppose that (X, A) is a pair of spaces and U is a subset of A such that the closure $\overline{U}$ is contained in the interior of A. Then the inclusion of pairs from $(X - U, A - U)$ to (X, A) induces isomorphisms in homology.*

(E.2) *Suppose that the space X can be written as a union of subsets $A \cup B$ such that the interiors of A and B form an open covering of X. Then the inclusion of pairs from $(B, A \cap B)$ to $(X = A \cup B, A)$ induces isomorphisms in homology.*

In particular, this axiom applies if A and B are open subsets of X.

AS noted in Eilenberg and Steenrod, the preceding axioms imply the existence certain long exact sequences known as *Mayer-Vietoris* sequences. These may be viewed as analogs of the Seifert-van Kampen Theorem, which describes the fundamental group of a space X in terms of the fundamental groups of two open subspaces U and V such that $X = U \cup V$ and all the spaces X, U, V, $U \cup V$, $X = U \cap V$ are nonempty and arcwise connected. If X is the union of two open subsets U and V (with not restrictions involving arcwise connectedness), these Mayer-Vietoris sequences exhibit a corresponding relationship involving the homology groups of U, V, $X = U \cup V$ and $U \cap V$. Once again, to avoid lengthy digressions we shall assume the existence of such sequences.

(E.3) *Let X be a topological space with $X = U \cup V$ such that* $\mathrm{Interior}(U) \cup \mathrm{Interior}(V) = X$. *Denote the inclusions of U and V in X by i_U and i_V respectively, and denote the inclusions of $U \cap V$ in U and V by g_U and g_V respectively. Then there is a long exact sequence*

$$\cdots \to H_{q+1}(X) \to H_q(U \cap V) \to H_q(U) \oplus H_q(V) \to H_q(X) \to \cdots$$

in which the map from $H_*(U) \oplus H_*(V)$ to $H_*(X)$ is given on the summands by $(j_U)_*$ and $(j_V)_*$ respectively, the map from $H_{q+1}(X)$ to $H_q(U \cap V)$ is the map Δ in the axiomatic data, and the map from $H_*(U \cap V)$ to $H_*(U) \oplus H_*(V)$ is given in coordinates by $(i_U)_*$ and $-(i_V)_*$ respectively (note the signs!!).

As before, this axiom applies if U and V are open subsets of X.

As in earlier discussions, let $X_i = U_i \cup V_i$, where $i = 1, 2$ and U_i, V_i satisfy the condition in (E.3). If $f : X_1 \to X_2$ is a continuous mapping such that $f[U_1] \subset U_2$ and $f[V_1] \subset V_2$ (we shall call this a map of triads), then we want this sequence to have good naturality properties with respect to f:

(E.4) *In the setting of the preceding paragraph, assume we are given a map of triads f from $(X_1; U_1, V_1)$ to $(X_2; U_2, v_2)$. Then there is a commutative ladder as below in which the horizontal lines represent the long exact Mayer-Vietoris sequences of* (E.3) *and the vertical maps are all induced by f:*

$$\begin{array}{ccccccccccc} \cdots & \to & H_{q+1}(X_1) & \to & H_q(U_1 \cap V_1) & \to & H_q(U_1) \oplus H_q(V_1) & \to & H_q(X_1) & \to & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \to & H_{q+1}(X_2) & \to & H_q(U_2 \cap V_2) & \to & H_q(U_2) \oplus H_q(V_2) & \to & H_q(X_2) & \to & \cdots \end{array}$$

In analogy with the naturality properties of (B.2) and (B.3), axiom (E.4) is a fairly straightforward consequence of (E.3) and (A.4).