ASSIGNMENT 1 – SOLUTIONS

1. We are given that $\pi_1(L) \cong \mathbb{Z}_{60}$. Therefore equivalence classes of connected covering spaces $M \to L$ are in 1–1 correspondence with subgroups of $\mathbb{Z}_{60}$.

If there were inequivalent covering spaces $M \to L$ and $M' \to L$ such that M and M' were homotopy equivalent, then $\pi_1(M) \cong \pi_1(M')$ but $\pi_1(M)$ and $\pi_1(M')$ are different subgroups of $\pi_1(L)$. ~~Thus~~ By standard results on cyclic groups and their subgroups, $\pi_1(M)$ and $\pi_1(M')$ would be cyclic subgroups of $\mathbb{Z}_{60}$ with the same order. On the other hand, there is ~~only~~ at most one subgroup of $\mathbb{Z}_{60}$ with a given order, and hence M homotopy equivalent to $M' \iff \pi_1(M) = \pi_1(M') \leq \mathbb{Z}_{60}$ so that $M \to L$ and $M' \to L$ are equivalent.

The subgroups of $\mathbb{Z}_{60}$ have the form $\mathbb{Z}_d$ where $d \mid 60$. Such a d has the form $2^a \cdot 3^b \cdot 5^c$ where $a \in \{0,1,2\}$ and $b, c \in \{0,1\}$. There are 12 possibilities for (a,b,c) and hence there are 12 equivalence classes of coverings. $\boxtimes$

2. By the Seifert-van Kampen Thm. we have a pushout diagram

$$\begin{array}{ccc} \pi_1(U \cap V) & \longrightarrow & \pi_1(U) \\ \downarrow & & \downarrow \\ \pi_1(V) & \longrightarrow & \pi_1(X) \end{array}$$

and by the uniqueness properties of such diagrams it will suffice to prove that if we are given $\gamma : A \xrightarrow{\cong} C$ and $\beta : A \to B$, then the diagram

$$\begin{array}{ccc} A & \xrightarrow[\cong]{\gamma} & C \\ \beta \downarrow & & \downarrow \beta \circ \gamma^{-1} \\ B & \xrightarrow{id} & B \end{array}$$

is a pushout.

Notice that it commutes by construction.

Suppose we are given a pair of homomorphisms $p : C \to K$, $q : B \to K$ such that $p \circ \gamma = q \circ \beta$.

Then $q : B \to K$ satisfies $q \circ id_B = q$ and $q \circ (\beta \circ \gamma^{-1}) = q \circ \beta \circ \gamma^{-1} = p \circ \gamma \circ \gamma^{-1} = p$, so there exists a homomorphism as in the Universal Mapping Property. To prove uniqueness, suppose $\varphi : B \to K$ is such that $\varphi \circ id_B = q$ and $\varphi \circ (\beta \circ \gamma^{-1}) = p$. Then $\varphi = q$ by the first equation, so the homomorphism $B \to K$ is unique. Therefore we have a pushout diagram. ∎