# Mathematics 205C, Spring 2011, Assignment 2

This will be due on **Monday, June 6, 2011, at 9:00 A.M.** at the beginning of the final examination. If you wish to use some version of TEX in writing up your answers, please feel free to do so. *You must show the work behind or reasons for your answers.*

Each problem is worth 20 points, and the values of the extra credit sections are noted individually.

**1.** If $U$ is an open subset in $\mathbb{R}^n$, then one can prove that $U = \cup_m K_m$, where $\{K_m\}$ is an increasing sequence of compact subsets and each $K_m$ is a polyhedron and if $x \in U - K_m$ then the distance from $x$ to $K_m$ is less than $2^{-m}\sqrt{n}$ (this is related to a basic result in measure theory which shows that $U$ is a countable union of nonoverlapping hypercubes). Assuming this and the axioms for homology, prove that $H_q(U) = 0$ if $q > n$. [*Hint:* Why is every compact subset of $U$ contained in some $K_m$?]

*Note:* In fact, one also can prove that $H_n(U) = 0$ but this requires more machinery than we have developed in this course.

Extra credit. (10 *points*). If $U$ is as above, prove that $H_q(U)$ is countable. [*Hint:* Let $L$ be the disjoint union of the groups $H_q(K_m)$. Why is $L$ countable, and why is there a surjection from $L$ to $H_q(K_m)$?]

*Note:* The groups $H_q(U)$ are not necessarily finitely generated. For example, $H_q(U)$ is a free abelian group on infinitely many generators if $U = \mathbb{R}^{q+1} - A$, where $A$ is the set of all points of the form $(k, 0, \cdots, 0)$, with $k$ running through the nonnegative integers.

**2.** Let $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ be the standard unit vectors in $\mathbb{R}^3$, and define maps $\alpha : \mathbb{R} \to \mathbb{R}^3$, $\beta : \mathbb{R}^3 \to \mathbb{R}^3$, $\gamma : \mathbb{R}^3 \to \mathbb{R}$ as follows:

$$\begin{aligned} \alpha(t) &= t\,\mathbf{i}\,, \qquad \text{(scalar product)} \\ \beta(\mathbf{v}) &= \mathbf{i} \times \mathbf{v}\,, \qquad \text{(cross product)} \\ \gamma(\mathbf{v}) &= \mathbf{i} \cdot \mathbf{v}\,, \qquad \text{(dot product)} \end{aligned}$$

Prove that the sequence

$$0 \longrightarrow \mathbb{R} \longrightarrow \mathbb{R}^3 \longrightarrow \mathbb{R}^3 \longrightarrow \mathbb{R} \longrightarrow 0$$

given by $\alpha - \beta - \gamma$ is an exact sequence (of real vector spaces).

Extra credit. (5 *points*). Explain why the result remains valid if we replace $\mathbf{i}$ by an arbitrary unit vector $\mathbf{u}$. [*Hint:* Look at an orthonormal basis for $\mathbb{R}^3$ whose first vector is $\mathbf{u}$ and whose third vector is the cross product of the first two.]

**3.** Suppose that $A$ and $B$ are disjoint simple closed curves in $S^2$. Prove that $S^2 - (A \cup B)$ has exactly three (arc) components and $H_1\big(S^2 - (A \cup B)\big) \cong \mathbb{Z}$.

Extra credit. (15 *points*). Prove that we can label the components $U, V, W$ such that $H_1(U) \cong \mathbb{Z}$ and $H_1(V) \cong H_1(W) \cong 0$. [*Hint:* If two of the components have nonzero homology, explain why they correspond to nonzero additive subgroups of $\mathbb{Z}$ whose intersection is the zero subgroup. Using the fact that every nonzero subgroup $S \subset \mathbb{Z}$ is the set of all integral multiples of some least positive element $d$, prove that the intersection $S_1 \cap S_2$ of two nonzero subgroups is nonzero.]