# Note on Problem 1

This problem contains an assertion (or assumption) that an open subset $U \subseteq \mathbb{R}^n$ is an increasing union of compact subsets $K_m$ such that if $x \in U - K_m$ then $\text{distance}(x, \mathbb{R}^n - U) < \frac{\sqrt{n}}{2^m}$. We shall sketch a proof of this when $n = 2$; similar ideas work in higher dimensions.

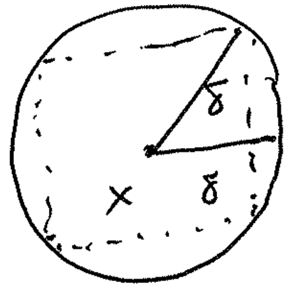
— x —

Here is the key observation:

PROPOSITION. Let $x \in U$ be open in $\mathbb{R}^2$, and suppose that $\text{distance}(x, \mathbb{R}^2 - U) \geq \frac{\sqrt{2}}{2^m}$. Write $x = (x_1, x_2)$ and choose integers $a_1, a_2$ such that $\frac{a_i}{2^{m+1}} \leq x_i \leq \frac{a_i + 1}{2^{m+1}}$. Then $U$ contains the closed region defined by $\frac{a_i}{2^{m+1}} \leq u_i \leq \frac{a_i + 1}{2^{m+1}}$.

The contrapositive statement is that if $U$ does not contain this square region then distance $(x, \mathbb{R}^2 - U) < \frac{\sqrt{2}}{2^m}$.

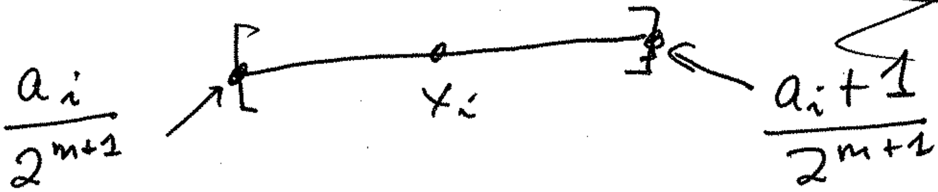
<u>PROOF OF PROPOSITION.</u> By hypothesis the open disk $N_{\sqrt{2}/2^m}(x)$ is contained in $U$. Hence the set of all $(p_1, p_2)$ such that $|p_i - x_i| < \frac{1}{2^m}$ is also contained in $U$.




The dotted line is the boundary of $|p_i - x_i| < \frac{\delta}{\sqrt{2}}$

(check this!).

Hence we have



$$x_i \le \frac{a_i+1}{2^{m+1}} \le x_i + \frac{1}{2^{m+1}} < x_i + \frac{1}{2^m}$$

$$x_i \ge \frac{a_i}{2^{m+1}} \ge x_i - \frac{1}{2^{m+1}} > x_i - \frac{1}{2^m}.$$

so that if $p_i \in \left[\frac{a_i}{2^{m+1}}, \frac{a_i+1}{2^{m+1}}\right]$ then

$p = (p_1, p_2) \in U$. $\blacksquare$

<u>Construction of $K_m$.</u> Take it to be the union of all square regions

$$\left[\frac{c_1}{2^{m+1}}, \frac{c_1+1}{2^{m+1}}\right] \times \left[\frac{c_2}{2^{m+1}}, \frac{c_2+1}{2^{m+1}}\right]$$

which are contained in $U$. $\blacksquare$