Assignment 2 - Solutions

1. We shall first prove the statement in the hint: If $K \subseteq U$ is compact, then distance $(K, \mathbb{R}^n - U) = \delta > 0$ for some δ. Choose m so large that

$$\frac{\sqrt{n}}{2^m} < \delta. \text{ Then } x \in U - K_m \Rightarrow$$

$$d(x, \mathbb{R}^n - U) < \frac{\sqrt{n}}{2^m} < \delta = \min_{y \in K} d(y, \mathbb{R}^n - U).$$

Hence $K \cap U - K_m = \phi$, so that $K \subseteq K_m$.

Now let $u \in H_q(U)$ where $q > n$, and let $K \subseteq U$ be compact such that $u \in$ Image $H_q(K) \to H_q(U)$. Then $K \subseteq K_m \subseteq U$ for some m by the preceding paragraph, so that $u \in$ Image $H_q(K_m) \longrightarrow H_q(U)$. Since K_m is an n-dimensional polyhedron, the groups $H_q(K_m) = 0$, so that u is the image of $0 \in H_q(K_n)$ and hence $u = 0$. ▪

<u>Extra Credit</u>. Again follow the hint. We have

$$L = \coprod_{m=1}^{\infty} H_q(K_m) \xrightarrow{\varphi} H_q(U).$$

and $q \geq 0$ is fixed.

Since each K_m is a polyhedron, each group $H_q(K_m)$ is countable and therefore L is too. Since the image of a countable set under a function is countable and φ is onto, it follows that $H_q(U)$ must be countable. ∎

2. We need to prove exactness of

$$0 \longrightarrow \underset{①}{\mathbb{R}} \xrightarrow{\alpha} \underset{②}{\mathbb{R}^3} \xrightarrow{\beta} \underset{③}{\mathbb{R}^3} \xrightarrow{\gamma} \underset{④}{\mathbb{R}} \longrightarrow 0$$

at ①-④ on a case by case basis.

① We only need to show $\alpha(t) = t\vec{\imath}$ is 1-1; but this is true because $\vec{\imath} \neq \vec{0}$.

② $\beta\alpha(t) = \vec{\imath} \times t\vec{\imath} = \vec{0}$, so Kernel $\beta \supseteq$ Image α. Note that rank$(\alpha) = 1$. On the other hand,

$$3 = \dim \mathbb{R}^3 = \text{rank}(\beta) + \text{nullity}(\beta),$$

and $\vec{\imath} \times \vec{\imath} = 0$, $\vec{\imath} \times \vec{\jmath} = \vec{k}$, $\vec{\imath} \times \vec{k} = -\vec{\jmath} \implies$ rank$(\beta) = 2$, so nullity$(\beta) = 1$ and Kernel β is 1-dimensional. Since Image$(\alpha) \subseteq$ Kernel(β) is 1-dim, these subspaces must be equal.

③ $\gamma\beta(\vec{v}) = \langle \vec{i}, \vec{i}\times\vec{v} \rangle = \langle \vec{i}\times\vec{i}, \vec{v} \rangle =$
$\langle \vec{0}, \vec{v} \rangle = 0$, so Image $\beta \subseteq$ Kernel γ.

In ② we showed Image β is 2-dimensional.
Since Kernel $\gamma = \text{Span}(\vec{j}, \vec{k})$ is also 2-dim,
these subspaces must be equal.

④ $\gamma(t\vec{i}) = \langle \vec{i}, t\vec{i} \rangle = t \Rightarrow \gamma$ is onto. ∎

EXTRA CREDIT. Follow the hint and let
$\vec{u}, \vec{v}, \vec{u}\times\vec{v} = \vec{w}$ be the given orthonormal basis.
Then α is still 1-1 and γ is still onto.
Also, Image$(\beta) = \text{Span}(\vec{v}, \vec{w}) \Rightarrow$
Kernel $(\beta) = \text{Span}(\vec{u}) = \text{Image}(\alpha)$
(once again $\beta\circ\alpha = 0$, so Kernel$(\beta) \supseteq$ Image(α),
and rank$(\beta) = 2 \Rightarrow \dim$ Kernel$(\beta) = 1 = \dim$ Image(α)).
These yield exactness at ①, ② and ④. To verify
exactness at ③, note that Kernel$(\gamma) = \text{Span}(\vec{v}, \vec{w})$,
which we know is Image(β). ∎

3. Use a relative Mayer-Vietoris sequence.

$$\tilde{H}_q(S^2-A) \cong \tilde{H}_q(S^2-B) \cong \begin{cases} \mathbb{Z} & q=0 \\ 0 & \text{otherwise} \end{cases}$$

(by the Jordan Curve Theorem). Thus we have (since $S^2-A \cup S^2-B = S^2-(A\cap B) = S^2$)

$$\begin{array}{ccccccc}
H_1(S^2) & \longrightarrow & \tilde{H}_0(S^2-A\cup B) & \longrightarrow & \begin{array}{c}\tilde{H}_0(S^2-A)\\ \oplus \\ \tilde{H}_0(S^2-B)\end{array} & \longrightarrow & \tilde{H}_0(S^2) \\
\parallel & & \parallel & & & & \parallel \\
0 & & (S^2-A)\cap(S^2-B) & & \mathbb{Z}\oplus\mathbb{Z} & & 0 \\
 & & ? & & & &
\end{array}$$

so that $\tilde{H}_0(S^2-(A\cup B)) \cong \mathbb{Z}\oplus\mathbb{Z}$ and hence $S^2-(A\cup B)$ has three (are) components. Also, we have

$$\begin{array}{ccccccc}
\begin{array}{c}H_2(S^2-A)\\ \oplus \\ H_2(S^2-B)\end{array} & \longrightarrow & H_2(S^2) & \longrightarrow & H_1(S^2-A\cup B) & \longrightarrow & \begin{array}{c}H_1(S^2-A)\\ \oplus \\ H_1(S^2-B)\end{array} \\
\parallel & & & & & & \parallel \\
0 & & \mathbb{Z} & & ? & & 0
\end{array}$$

which implies $H_1(S^2-(A\cup B)) \cong \mathbb{Z}$. $\blacksquare$

EXTRA CREDIT. If U', V', W' are the connected components of $S^2 - A \cup B$ we have

$$\mathbb{Z} \cong H_1(U' \cup V' \cup W') = H_1(U') \oplus H_1(V') \oplus H_1(W')$$

so it is enough to show that if $\mathbb{Z} \cong G \oplus H$ for abelian groups G, H, then either $G = 0$ or $H = 0$ (by induction, if $\mathbb{Z} \cong \oplus G_i$, then all but one subgp G_i are zero).

But if $\mathbb{Z} \cong G_1 \oplus G_2$ with G_1, G_2 nonzero, then $\mathbb{Z}$ contains subgroups $H_i \leftrightarrow G_i$ such that $H_1, H_2 \neq 0$ but $H_1 \cap H_2 = 0$. Now there are $d_i > 0$ such that $H_i = \langle d_i \rangle$, all multiples of d_i. Therefore $H_1 \cap H_2$ contains the nonzero element $d_1 d_2$; in other words $0 \neq H_1, H_2 \subseteq \mathbb{Z} \Rightarrow H_1 \cap H_2 \neq 0$, so $\mathbb{Z}$ cannot be written as a nontrivial direct sum. If we apply this to $H_1(U' \cup V' \cup W') \cong \mathbb{Z}$, this means one of $H_1(U'), H_1(V'), H_1(W') = \mathbb{Z}$ and the other two must be zero. ∎