Comments on Problem 3

Several solutions used the following result:

Theorem. *Let $A \subset S^2$ be a simple closed curve, and let W be a component of $S^2 - A$. Then W is homeomorphic to an open 2-disk.*

This is in fact true, but a proof was not given in the course. One way of proving this result is to note that the methods of the course show that $H_1(W) = 0$ and to show that a connected region in the plane with this property is diffeomorphic (in fact, complex analytically equivalent) to an open disk by the Riemann Mapping Theorem as is done in Ahlfors, *Complex Analysis*. One can go even further and prove that $\overline{W} = W \cup A$ is homeomorphic to D^2 such that A corresponds to S^1, but this result (called the Schönflies Theorem) is much deeper.

Using the theorem, one can proceed (as in several solutions) to say that B lies in one component of $S^2 - A$, so say that the latter is given as a union $U \cup V$ of its components and assume (without loss of generality) that $B \subset U$. This implies that $U - B$ is a union of components $W_1 \cup W_2$. If we use the identification of the one point compactification $U^\bullet$ with S^2, then we know that the point at infinity lies in the closure of one of these components, say W^2; let W^* be this closure, which is an open subset of $U^\bullet$. Then by the preceding we know that W_1 and W^* are homeomorphic to open disks, and hence W_2, which is the complement of a point in the topological open disk W^*, must be homeomorphic to $S^1 \times (0, 1)$. [*Note:* In many papers this step was omitted.] This argument proves the assertion about three components in the complement of $A \cup B$ and also the fact that H_1 of one component is infinite cyclic while H_1 of the remaining components must vanish (which is the extra credit statement).

Since the version of the Riemann Mapping Theorem mentioned above may have been covered in a complex analysis course like 210B, full credit was given for solutions as above (provided nothing was left out!).