Spaces with finite fundamental groups

Theorem. Let X be a simply connected, locally arcwise connected, T_2 space. Let G be a finite group, and assume that G acts on the right of X by homeomorphisms. Assume the action is FREE $\boxed{xg = x \text{ some } (x,g) \Longleftrightarrow g = 1}$. Then the quotient space projection $X \to X/G$ is a covering space projection, the space X/G is T_2, and $\pi_1(X/G, pt.) \cong G$.

Examples. $X = S^n$, $G = \{\pm 1\}$, so $X/G \cong \mathbb{RP}^n$ (real projective n-space).

<u>Lens spaces</u> $X = S^{2n-1}$, G = finite cyclic group acting by complex mult. ($G \cong n$-th roots of unity where $n = |G|$).

<u>Nonabelian π_1</u> S^3 = unit quaternions

$1, i, j, k$ with mult. table

$$i^2 = j^2 = k^2 = -1, \quad ij = -ji = k$$
$$jk = -kj = i, \quad ki = -ik = j$$

$\mathbb{K}$

Q = gp of order 8 with elts $\pm 1, \pm i, \pm j, \pm k$.

$\pi_1(S^3/Q) \cong Q$.

~~Arbitrary finite G~~ ~~X, Y compact subsets~~

some

<u>COROLLARY.</u> Every finitely generated abelian group is π_1 of some compact metric space.

<u>Proof of Theorem</u> Key point is

<u>Lemma</u> If $x \in X$ then there is an open neighborhood U of x such that $g_1 \neq g_2 \Rightarrow g_1 U \cap g_2 U = \phi$. Likewise, if x and y in X are such that $y \neq \cancel{gx}\, xg$ for all x, then there are open neighborhoods U of x and V of y such that for all $g_1, g_2 \in G$ we have $g_1 U \cap g_2 V = \phi$.

<u>PROOF OF LEMMA.</u> Since X is T_2, if $x \in X$ and $g \neq 1$ there are disjoint open sets $W(g)$ and $W'(g)$ containing x and $\cancel{gx}\, xg$ respectively. Let

$$U = \Big(\bigcap_{g \neq 1} W(g)\Big) \cap \Big(\bigcap_{g \neq 1} W'(g)\, g^{-1}\Big)$$

which is an open neighborhood of x.

If $1 \neq g_0$, then we have

$$V \cap U \cdot g_0 \subseteq W(g_0) \cap W'(g_0) \underbrace{g_0^{-1} g_0}_{=1} = \phi.$$

One can find a similar V for y, and we can make it so small that $V \cap U \cdot g = \phi$ for all $g \in G$. Thus we have $V \cdot g_1 \cap U \cdot g_2 = V \cdot g_1 \cap U \cdot g g_1$ where $g = g_2 g_1^{-1}$, and this intersection must be empty as desired.