# Classifying Coverings

$p : (X, x_0) \longrightarrow (Y, y_0)$ coverings (usual hypotheses)

$q : (W, w_0) \longrightarrow (Y, y_0)$

are <u>equivalent</u> if there is a homeomorphism $h : (W, w_0) \longrightarrow (X, x_0)$ such that

$p \circ h = q$.

<u>Prop.</u> If coverings are equivalent, they determine the same subgroup of $\pi_1(Y, y_0)$.

Proving a converse is harder.

$X$ is semilocally simply connected if it is locally arcwise connected $T_2$ and every point has at least one simply connected neighborhood.

<u>Thm.</u> If $X$ is connected + semilocally simply connected, then $X$ has a simply connected covering space.

## Motivation for the construction

Suppose a simply connected covering space exists. $p : (\tilde{Y}, \tilde{y}_0) \longrightarrow (Y, y_0)$. Note that the following are true:

1. There is a 1-1 correspondence between points of $\tilde{Y}$ and endpoint preserving homotopy classes of curves in $Y$ which start at $y_0$.

2. Suppose $p(z) = y$ and $y \in U$, where $U$ is open and evenly covered; let $z \in V$ open (in $\tilde{Y}$) s.t. $p$ maps $V$ homeomorphically to $U$, ~~then~~ and $U, V$ are arcwise connected. Then the points of $V$ correspond to curves of the form $\alpha + \beta$ where $\alpha$ corresponds to $z$ as in 1 and $\beta$ is a curve in $U$ which starts at $y$.

The arguments simplify a bit if we assume that every point in $Y$ has a neighborhood base of open simply connected sets.

In the construction, one starts by defining points of $\widetilde{Y}$ as in (1) and neighborhoods somewhat as in (2).

The steps in the proof are noted very explicitly in Section 82 of Munkres, <u>Topology</u> (pp. 494-498). We shall only comment on the proof that $\widetilde{Y}$ is simply connected.

By construction, $F = p^{-1}[\{y_0\}]$ is just $\pi_1(Y, y_0)$, and in fact the action map $F \times \pi_1(Y, y_0) \longrightarrow F$ is just multiplication.

If $\pi_1(\tilde{Y}, \tilde{y}_0) = H \neq 1$, then $h \in H$ and $z \in F \Longrightarrow zh = z$. But this is not the case.

The preceding yields

CLASSIFICATION THEOREM. Suppose $Y$ is semilocally simply connected. ~~and~~ Then there is a 1-1 correspondence between equivalence classes of based covering space projections $p: (X, x_0) \longrightarrow (Y, y_0)$ and subgroups of $\pi_1(Y, y_0)$.

Most of this is known, but we need to show that every subgroup arises. But given $H \subseteq \pi_1(Y, y_0)$, we can take $X_H = \tilde{Y}/H$ ~~to~~ and check this is a covering with $\pi_1(X_H) \cong H$.