Covering Spaces and Smooth Structures

Let $(M, \mathcal{S})$ be a smooth manifold with smooth structure $\mathcal{S}$, and let $p: N \to M$ be a covering space projection such that N is only known to be a topological manifold. Then there is a smooth structure $p^*\mathcal{S}$ on N which is characterized up to diffeomorphism by the following:

Suppose that (U, h) is a smooth connected chart for M such that $h[U] \subseteq M$ is evenly covered, and let $U' \subseteq p^{-1}[h[U]]$ be such that $p|U'$ is a homeomorphism. Then the unique lifting $p|U'$ maps U' diffeomorphically onto $h[U]$.

The proof is left as an exercise; one can use the characterizing property to construct the charts on N.

<u>RELATED EXERCISE.</u> In terms of functional structures, one can also characterize the smooth structure on N as follows: If $V \subseteq N$ is evenly covered and $V' \subseteq M$ is such that $p|V'$ maps V' homeomorphically to V, then composition with p defines an isomorphism of smooth function algebras from $\mathcal{F}(V)$ to $\mathcal{F}(V')$.

Here is a typical consequence of these observations and the previous section:

<u>PROP.</u> Suppose that $U \subseteq \mathbb{R}^n$ is open and connected. Then ~~it~~ has a smooth covering space $\tilde{U} \to U$ with $\pi_1(\tilde{U}, \cdot) = \{1\}$.

<u>REMARKS.</u> 1. If $n=2$, then the Riemann Mapping Theorem from complex analysis implies that $\tilde{U}$ is diffeomorphic to $\mathbb{R}^2$. $\boxed{U \cong S^2 \times \mathbb{R}}$ However, this does not extend to $n \geq 3$.

2. One might also ask if U open in $\mathbb{R}^n$ $(n \geq 2)$ $\Rightarrow$ $\tilde{U}$ is diffeomorphic to an open subset in $\mathbb{R}^n$. The answer is apparently not known.