Computing Fundamental Groups

Many (probably most) techniques for computing algebraic invariants like fundamental groups are most effective for spaces which are built using relatively simple pieces.

Seifert - van Kampen Theorem

$X = U \cup V$ s.t. U, V open & arcwise connected AND $U \cap V \neq \emptyset$ is also arcwise connected. Then $p \in U \cap V$

① $\pi_1(X, p)$ is generated by the images of $\pi_1(U, p)$ and $\pi_1(V, p)$.

Given groups H_1 & H_2, their free product is the "most general" group S such that

S is generated by subgroups isomorphic to H_1 and H_2. The nontrivial elts are basically uniquely expressible as products $a_1 b_1 \dots a_m b_m$ with $a_i \in H_1$, $b_i \in H_2$. All the elements are nontrivial except possibly a_1 or b_m.

$S = H_1 * H_2$

② The kernel of the associated map

$$\pi_1(U, p) * \pi_1(V, p) \longrightarrow \pi_1(\underbrace{U \cup V}_{X=}, p)$$

is generated by classes of the form $(j_U * y)(j_V * y)^{-1}$, where

$$j_U : U \cap V \longrightarrow U$$
$$j_V : U \cap V \longrightarrow V$$

are inclusions

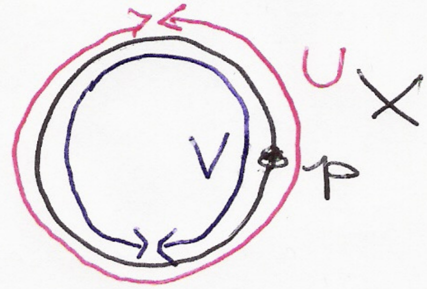
Notes. 1. General considerations show that the kernel in ② contains the given elements and subgroup.

2. The conclusion in ① is not necessarily true if $U \cap V$ is not connected.

$X = S^1$

$U = S^1 - \{(0,1)\}$

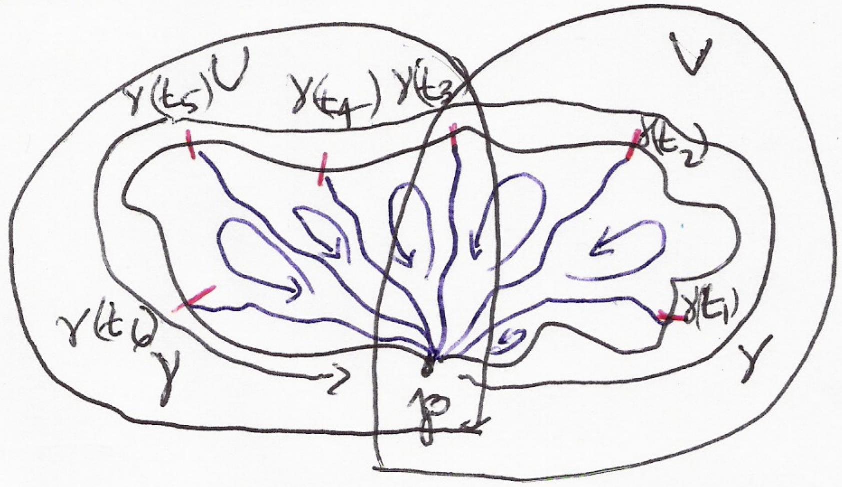
$V = S^1 - \{(0,-1)\}$



$\pi_1(U,p) \cong \pi_1(V,p) \cong 1$ but $\pi_1(S^1,p) \neq \{1\}$.

Proving ① is easier than proving ②.

PROOF OF ①



5.4

① Partition $[0,1]$ into pieces so that γ maps each piece into U or V.

② Find curves α_i joining p to $\gamma(t_i)$ s.t.

α_i lies in U if $\gamma(t_i) \in U$

α_i —"— V —"— $\in V$

α_i —"— $U \cap V$ —"— $\in U \cap V$.

α_0 = constant

③ Let γ_i = piece from $\gamma(t_{i-1})$ to $\gamma(t_i)$

Check

$$\gamma \sim \gamma_1 + \dots + \gamma_m =$$

$$\underbrace{\gamma_1 - \alpha_1} + \underbrace{\alpha_1 + \gamma_2 - \alpha_2} \dots + \underbrace{\cancel{\;}\alpha_{m-1} + \gamma_m}.$$

closed curves.

Note that $\alpha_{i-1} + \gamma_i - \alpha_i$ lies in either U or V.

Proofs of (2) are in Bredon and the course directory (svkproofs.pdf). The latter uses covering spaces and assumes semilocal simple connectivity in some form.

<u>Note</u>. The group $\pi_1(X,p)$ is an example of a pushout construction. In other words we have a gp diagram

$$\begin{array}{ccc} K & \xrightarrow{j_1} & H_1 \\ {\scriptstyle j_2}\downarrow & & \downarrow{\scriptstyle g_1} \\ H_2 & \xrightarrow{g_2} & G \end{array}$$

s.t. (i) the images of $g_1 + g_2$ generate G

(ii) if $f_1, f_2 : H_1, H_2 \longrightarrow \Gamma$ are group homomorphisms s.t. $f_1 \circ j_1 = f_2 \circ j_2$, then there is a unique $f : G \longrightarrow \Gamma$ s.t. $g_i \circ f = f_i$ $(i=1,2)$.

UNIVERSAL MAPPING PROPERTY

<u>Consequences</u> 1. U & V simply connected $\Longrightarrow$ so is $U \cup V$.

2. $S^n = (S^n - \{e_n\}) \cup (S^n - \{-e_n\})$

$(S^n - \{e_n\}) \cap (S^n - \{-e_n\}) \cong$

$S^{n-1} \times \mathbb{R} \Longrightarrow S^n$ simply connected

if $n \geq 2$. (quick proof)

3. U_1 & U_2 open in $\mathbb{R}^n$; $U_1 \cap U_2$ nonempty convex

$\Longrightarrow \pi_1(U_1 \cup U_2) = \pi_1(U_1) * \pi_1(U_2)$.

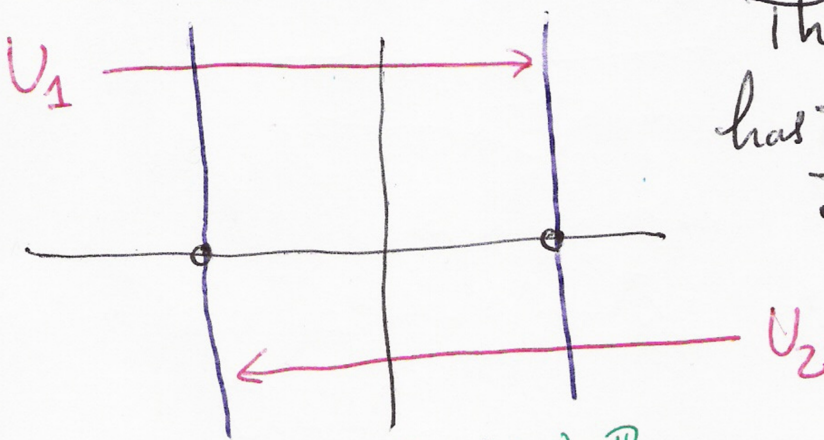
Suppose $n=2$

$U_1 = (\mathbb{C} - \{-1\}) \cap \{x < 1\}$

$U_2 = (\mathbb{C} - \{1\}) \cap \{x > -1\}$

$\mathbb{C} - \{\pm 1\}$ $\|$

Then $U_1 \cup U_2$ has $\pi_1 = \mathbb{Z} * \mathbb{Z}$ free group on two generators.



$U_1 \cap U_2 = (-1,1) \times \mathbb{R}$ is convex

Suppose now that $n \geq 3$ and

$X = \mathbb{R}^n - \{\pm e_1\}$, $\vec{e}_1 = (1, 0, 0, \ldots \text{etc})$

Define $U_1 = \{\vec{x} \in \mathbb{R}^n \mid x \neq \vec{e}_1, x_1 > -1\}$

$U_2 = \{x \in \mathbb{R}^n \mid x \neq -\vec{e}_1, x_1 < 1\}$

Then U_1, U_2 and $U_1 \cap U_2$ are simply connected, and therefore $U_1 \cup U_2 = X$ is too.

<u>GENERALIZATION.</u> Extend these to $\mathbb{R}^n -$ k points which lie on the line joining $\vec{0}$ to $\vec{e}_1$. The results split into cases where $n = 2$ and $n \geq 3$.

~~Exercise: Generalize to $\mathbb{C} - n$ points on the real axis!~~

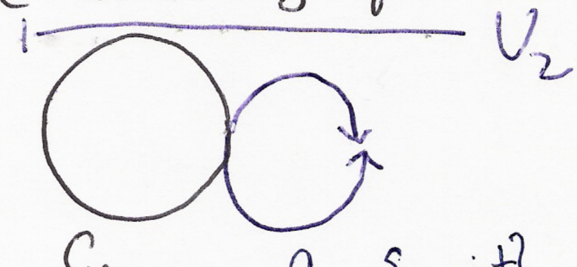
4. $X =$ one point union of two circles

C_1, C_2. $\quad C_1 \cap C_2 = \{p\}$.

$U_1 = X - \{q_1\} \quad q_1 \in C_1 - C_2$

$U_2 = X - \{q_2\} \quad q_2 \in C_2 - C_1$.

$C_1 \subseteq U_2$
$C_2 \subseteq U_1$ are homotopy equivalences



$U_1 \cap U_2$ is
are conn. (why?)

So also $\pi_1(X) \cong \mathbb{Z} * \mathbb{Z}$.

Generalize to one pt union of three circles,
or of any finite # of circles

5. $X = S^2 \cup D^2$

(Sketch)

$U_1 = \{x_3 > -\frac{1}{2}\}$

$U_2 = \{x_3 < \frac{1}{2}\}$

$U_1 \cap U_2 =$ "tropical zone" on S^2 plus D^2.

D^2 is a deformation retract of $U_1 \cap U_2$

(shrink "tropical" zone on sphere to equator)

$V_1 = \{x_3 > 0\}$

$V_2 = \{x_3 < 0\}$

$\Rightarrow U_i = V_i \cup (U_1 \cap U_2)$

$\pi_1(V_1), \pi_1(V_2), \pi_1(U_1 \cap U_2) = \{1\}$

Hence $\pi_1(U_1), \pi_1(U_2) = \{1\}$

Hence $\pi_1(X) = \{1\}$.