

Homology of simplicial complexes

(P, K) simplicial complex. Choose a linear ordering of the vertices. Represent a simplex by its vertices in order.

Chain groups $C_q(P, K, \text{ordering}) = C_q(P, K)$ for short.

$C_q(P, K) =$ free abelian group on the q -simplices $x_0 \dots x_q$.

Boundary:

$$d(x_0 \dots x_q) = \sum_{i=0}^q (-1)^i x_0 \dots \overset{\text{OMIT } x_i}{\wedge} \dots x_q =$$

$$\sum_{i=0}^q (-1)^i \partial_i(x_0, \dots, x_q).$$

\uparrow
 i th face (opposite the i th vertex)

Reasons for this:

$$d(x_0 x_1) = x_1 - x_0.$$

See ~~algebraic topology~~ chain boundary.pdf for additional motivation.

Fundamental identity:

$$d_{q-1} \circ d_q = 0, \text{ so}$$

$$\text{Kernel } d_{q-1} \supseteq \text{Image } d_q.$$

Homology groups: $H_q(P, K) = \frac{\text{Kernel } d_q}{\text{Image } d_{q+1}}.$

Kernel $d_q = \text{cycles}$, Image $d_{q+1} = \text{boundaries}$.

Generalizes earlier definition in the 1-dim case.

It is useful to define a very abstract version of simplicial chains:

Def. A chain complex over a ring R is a pair (C_*, d_*) where C_q ~~is an~~ ^{are} R -modules and $d_q: C_q \rightarrow C_{q-1}$ s.t. $d_{q-1} \circ d_q = 0$.

$$\text{Homology } H_q(C) = \frac{\text{Ker } d_q}{\text{Image } d_{q+1}}$$

Map of chain complexes:

$$f_q: C_q \rightarrow D_q \quad \text{s.t.}$$

$$\begin{array}{ccc} C_q & \xrightarrow{f_q} & D_q \\ d_q^C \downarrow & & d_q^D \downarrow \\ C_{q-1} & \xrightarrow{f_{q-1}} & D_{q-1} \end{array} \quad \text{commutes.}$$

$$\boxed{\begin{array}{l} f_{q-1} \circ d_q^C = \\ d_q^D \circ f_q \end{array}}$$

Example (Q, L) subcomplex (P, K)

[\Leftrightarrow if simplex $\sigma \in L$ then so are all its faces]. $Q \subseteq P$.

Then inclusion $C_*(Q, L) \rightarrow C_*(P, K)$ is a chain map (chain subcomplex).

Lemma $f_{\#}: C_{\#} \rightarrow D_{\#}$ chain map \Rightarrow

\exists homomorphisms ~~$f_{\#}: H_{\#}(C) \rightarrow H_{\#}(D)$~~

$$f_{\#}: H_{\#}(C) \rightarrow H_{\#}(D)$$

s.t. if $u \in C_q$ represents $x \in H_q(C)$, then $f_{\#}(u)$ represents $f_{\#}(x) \in H_q(D)$.

Quotient complex.

$$(A_{\#}, d_{\#}^A) \subseteq (B_{\#}, d_{\#}^B) \rightsquigarrow \exists \text{ chain map}$$

$$(B_{\#}, d_{\#}^B) \xrightarrow{\text{onto}} (B_{\#}/A_{\#}, d_{\#}^{B/A}).$$

Proof of Lemma Must show

$$\textcircled{1} \quad 0 = d_q x \implies 0 = d_q f(x) \quad [= f(d_q(x))]$$

$$\textcircled{2} \quad dx = dx' = 0 \text{ and } x - x' \implies dy = 0$$

$$\text{implies } df(y) = f(dy) = f(x) - f(x').$$

$\textcircled{3}$ module hom.

$$\begin{array}{l} x \text{ reps } u \\ x' \text{ reps } u' \\ c \in R \end{array} \implies \begin{array}{l} x+x' \text{ reps } u+u' \\ cx \text{ reps } cu. \end{array}$$

Further properties (functoriality)

$$(1_C)_* = \text{identity on } H_*(C)$$

$$(gf)_* = g_* f_*$$

[\implies if f is an iso, then so is f_*].

COMPUTING HOMOLOGY

Assume each C_n is fin gen free abelian,

$C_q = 0$ for all but finitely many q .

$R = \text{principal ideal domain}$

Also assume we are given ordered free bases B_q for C_q and that we can easily find the matrix for d_q w.r.t. B_q and B_{q-1} (clearly true for simplicial chains). Then there is a direct way of computing $H_q(C)$.

See [Computing \$H_q\$.pdf](#)

Back to the general form of Green's Theorem

In terms of the concepts presented, the assumptions behind the general form of Green's Theorem are:

(1) We can triangulate D so that its boundary corresponds to a subcomplex (the big cx is $2D$).

(2) If $\alpha_1, \dots, \alpha_M$ are the free generators of $C_2(P, K)$, then $\exists \varepsilon_i = \pm 1$ s.t.

$$d\left(\sum \varepsilon_i \alpha_i\right) = \gamma_0 - \sum_{j>0} \gamma_j$$

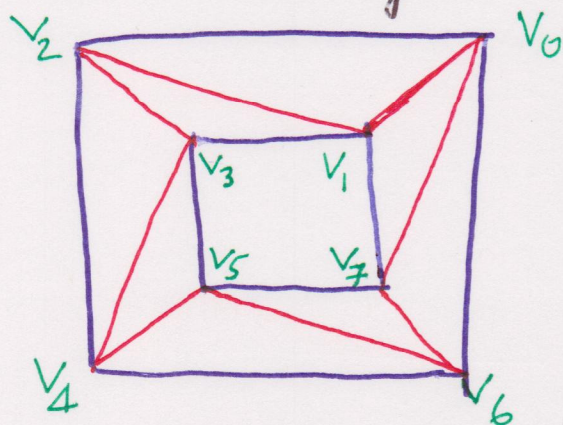
where $\gamma_j \in C_1(P, K)$ corresponds

to going around boundary curve

γ_j in the counterclockwise sense.

** Need the ε_i ^{bec app} ~~so that~~ the standard identification of Δ_2 with a triangle in K (order preserving on vertices) may have negative Jacobian.

Example Look at the region between two squares with the indicated triangulation



The counter clockwise inner and outer boundary curves are given by

$$V_1 V_3 + V_3 V_5 + V_5 V_7 - V_1 V_7 \quad (\text{inner})$$

~~$$V_2 V_4 + V_4 V_6 + V_6 V_0 - V_2 V_0$$~~

$$V_0 V_2 + V_2 V_4 + V_4 V_6 - V_0 V_6 \quad (\text{outer})$$

and (outer) - (inner) is the boundary of

$$- V_0 V_1 V_2 + V_1 V_2 V_3 - V_2 V_3 V_4 + V_3 V_4 V_5 \quad (\text{continued})$$

$$- V_4 V_5 V_6 + V_5 V_6 V_7 - V_0 V_6 V_7 + V_0 V_1 V_7$$

Finally, we note that the assumptions ① and ② can be verified, but this requires more machinery than we can develop in the present course.