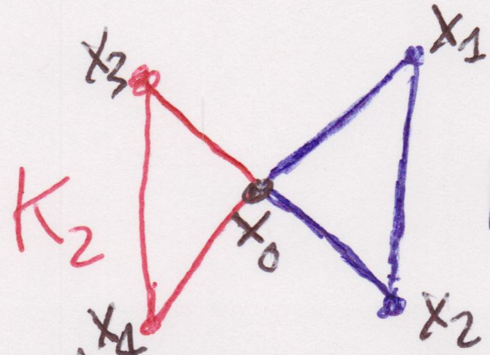
Analyzing homology groups using the homology of subcomplexes

Example



$K = K_1 \cup K_2$ (1-dim.)

$H_1(K_1 \cup K_2) = i_{1*} H_1(K_1) \oplus i_{2*} H_2(K_2)$

$K = K_1 \cup K_2$

$\{x_0\} = K_1 \cap K_2$

(in fact, this holds more generally and we shall prove it in the exercises).

General Question Describe $H_*(K_1 \cup K_2)$ in terms of $H_*(K_1), H_*(K_2)$ and $H_*(K_1 \cap K_2)$

Model Counting formula $\#(A \cup B) = \#(A) + \#(B) - \#(A \cap B)$, A & B finite sets.

There are two related methods involving long exact sequences.

① Suppose $K = K_0 \cup A$, where A is a simplex and $K_0 \cap A = \partial A$.

NOTE Every simplicial complex is built from a finite sequence of such constructions starting with some finite set of points.

Long exact homology sequence

$$\cdots H_{q+1}(K, K_0) \longrightarrow H_q(K_0) \longrightarrow H_q(K) \longrightarrow H_q(K, K_0) \cdots$$

Suppose $H_*(K_0)$ is known.

If we can compute $H_*(K, K_0)$, then maybe we can get some insight into $H_*(K)$.

More generally, suppose $K = K_0 \cup K_1$.

Prop. $C_*(K_1, K_1 \cap K_0) \xrightarrow{\cong} C_*(K, K_0)$.

$\boxed{\text{Excision}}$ $H_*(K_1, K_1 \cap K_0) \xrightarrow{\cong} H_*(K, K_0)$.

First line $\Rightarrow$ Second

Application Let $K_1 = A$ so that $K_0 \cap K_1 = \partial A$. Then $H_*(K, K_0) \cong H_*(A, \partial A)$, which we know: $\begin{cases} \mathbb{Z} & q = \dim A \\ 0 & \text{otherwise} \end{cases}$

Proof. $C_*(K_1)/C_*(K_1 \cap K_0) =$

$C_*(K_1)/C_*(K_1) \cap C_*(K_0) \xrightarrow[\text{ISOMORPHISM THM}]{\cong}$

$[C_*(K) = C_*(K_0) + C_*(K_1)]/C_*(K_0)$.

Note If $\begin{matrix} L' & \subseteq & K' \\ \cap & & \cap \\ L & \subseteq & K \end{matrix}$, then subcomplex inclusions yield a map of quotient complexes $C_*(K', L') \to C_*(K, L)$.

② Mayer-Vietoris long exact sequences

$$K = K_1 \cup K_2$$

$$\begin{tikzcd} K_1 \cap K_2 \arrow[r, "j_1"] \arrow[d, "j_2"'] & K_1 \arrow[d, "i_1"] \\ K_2 \arrow[r, "i_2"'] & K_1 \cup K_2 \end{tikzcd}$$

Prop. There is a short exact sequence of chain complexes

$$0 \longrightarrow C_*(K_1 \cap K_2) \longrightarrow \begin{matrix} C_*(K_1) \\ \oplus \\ C_*(K_2) \end{matrix} \longrightarrow C_*(K) \longrightarrow 0$$

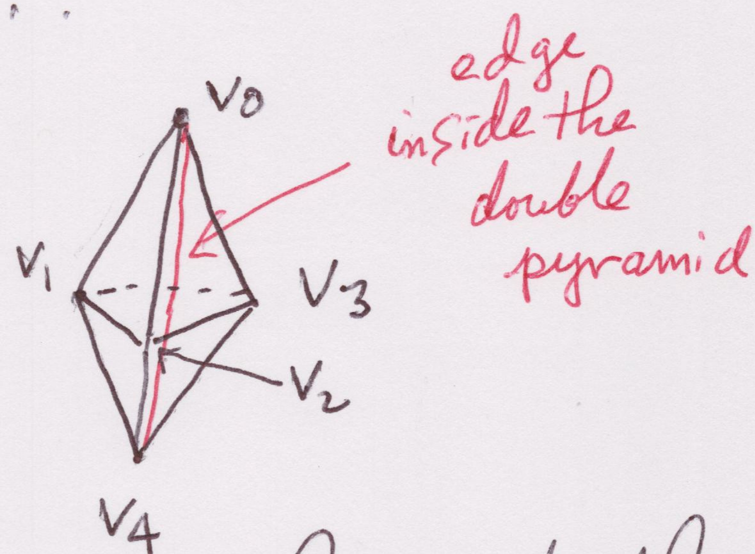
$$z \longrightarrow (j_{1\#} z, -j_{2\#} z) \qquad (a, b) \longrightarrow i_{1\#} a + i_{2\#} b$$

NOTE THE SIGN!!!

This yields the Mayer-Vietoris Exact Sequence:

$$H_{q+1}(K_1 \cup K_2) \longrightarrow H_q(K_1 \cap K_2) \xrightarrow{(j_{1*}, -j_{2*})} \begin{array}{c} H_q(K_1) \\ \oplus \\ H_q(K_2) \end{array} \xrightarrow{i_{1*} + i_{2*}} H_q(K_1 \cup K_2) \ldots$$

<u>Example</u>



Simplices are

$v_0 v_1 v_2$
$v_0 v_1 v_3$
$v_0 v_2 v_3$
$v_1 v_2 v_4$
$v_2 v_3 v_4$
$v_1 v_3 v_4$
$v_0 v_4$

and their faces.

Compute the homology.

<u>Steps</u> K_1 = upper pyramid faces.

$$K_1 = \overset{L_3}{v_0v_1v_2} \cup \overset{L_2}{v_0v_1v_3} \cup \overset{L_1}{v_0v_2v_3}.$$

Know H_* of each 2-simplex.

$L_1 \cap L_2$ = edge v_0v_3.

M-V sequence reduces to

$$\underset{H_q(L_1\cap L_2)}{\overset{}{0}} \longrightarrow \underset{H_q(L_1)\oplus H_q(L_2)}{0 \oplus 0} \longrightarrow H_q(L_1\cup L_2) \longrightarrow 0 \qquad \text{if } q \geq 2$$

so $H_q(L_1\cup L_2) = 0$ if $q \geq 2$.

Tail end

$$0\oplus 0 = H_1(L_1)\oplus H_1(L_2) \longrightarrow H_1(L_1\cup L_2) \xrightarrow{\Delta} H_0(L_1\cap L_2) = \mathbb{Z}$$

$$\downarrow \quad \text{1-1 map} \qquad 1 \mapsto (1,-1)$$

$$H_0(L_1)\oplus H_0(L_2) \quad \mathbb{Z}\oplus\mathbb{Z}$$

$$\downarrow \quad \text{connected}$$

$$H_0(L_1\cup L_2) \quad \mathbb{Z}$$

So $\Delta = 0$ and $H_1(L_1\cup L_2)$ is also trivial.

$$K_1 = (L_1 \cup L_2) \cup L_3, \quad (L_1 \cup L_2) \cap L_3 =$$

$$V_0V_2 \cup V_0V_1 \leftarrow$$ homology zero if $q=0$, $\mathbb{Z}$ if $q=0$.

As before, can use MV to show

$$H_q(K_1) = \begin{cases} \mathbb{Z} & q=0 \\ 0 & q>0. \end{cases}$$

A similar conclusion holds for $K_2 =$ lower pyramid faces.

$K_1 \cap K_2 =$ triangle $V_1V_2V_3$.

Compute $H_*(K_1 \cup K_2)$ by M–V:

$$\underbrace{\begin{matrix} H_{q+1}(K_1) \\ \oplus \\ H_{q+1}(K_2) \end{matrix}}_{0} \longrightarrow H_{q+1}(K_1 \cup K_2) \longrightarrow H_q(K_1 \cap K_2) \longrightarrow \underbrace{\begin{matrix} H_q(K_1) \\ \oplus \\ H_q(K_2) \end{matrix}}_{0}$$

isomorphism if $q>0$.

So $H_{q}(K_1 \cup K_2) = \begin{cases} \mathbb{Z} & r=2 \\ 0 & \text{otherwise for } r \geq 3. \end{cases}$

Now look at the tail end.

$$\begin{matrix} H_1(K_1) \\ \oplus \\ H_1(K_2) \end{matrix} \longrightarrow H_1(K_1 \cup K_2) \longrightarrow H_0(K_1 \cap K_2) \longrightarrow \begin{matrix} H_0(K_1) \\ \oplus \\ H_0(K_2) \end{matrix}$$

$$0 \longrightarrow \quad \xrightarrow{\Delta} \mathbb{Z} \longrightarrow \mathbb{Z} \oplus \mathbb{Z}$$

$$1 \rightsquigarrow (1,-1)$$

so $\Delta = 0$ and $H_1(K_1 \cup K_2) = 0$.

Let $M = K_1 \cup K_2$
$N = v_0 v_4$.

$K = M \cup N$
$M \cap N = \{v_0, v_4\}$.

What do we get this time?

Need only worry about $q = 0, 1, 2$ since other chain groups $= 0$.

$$H_2(M\cap N) \to \begin{matrix} H_2(M) \\ \oplus \\ H_2(N) \end{matrix} \xrightarrow{\varphi} H_2(M\cup N) \to$$

$$0 \longrightarrow \mathbb{Z}\oplus 0 \longrightarrow \boxed{?} \longrightarrow$$

$$H_1(M\cap N) \to \begin{matrix} H_1(M) \\ \oplus \\ H_1(N) \end{matrix} \longrightarrow H_1(M\cup N) \xrightarrow{\Delta}$$

$$0 \longrightarrow 0 \longrightarrow \boxed{?} \longrightarrow$$

$$H_0(M\cap N) \longrightarrow \begin{matrix} H_0(M) \\ \oplus \\ H_0(N) \end{matrix} \longrightarrow \begin{matrix} H_0(M\cup N) \\ = \mathbb{Z}. \end{matrix}$$

$$\mathbb{Z}\oplus\mathbb{Z} \xrightarrow{\theta} \mathbb{Z}\oplus\mathbb{Z} \xrightarrow[\text{COORDS}]{\text{ADD}} \mathbb{Z}$$

$$\begin{matrix}(1,0)\\(0,1)\end{matrix} \rightsquigarrow (1,-1)$$

φ must be an isomorphism, so $H_2(K) \cong \mathbb{Z}$. Kernel $\theta \cong \mathbb{Z}$ gen by $(1,-1)$.

Δ is 1-1 onto Kernel θ, so $H_1(K) = \mathbb{Z}$ also.