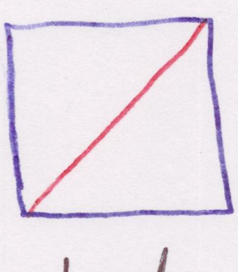
The next step(s)

Empirical considerations

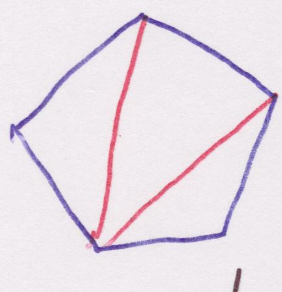
Regular polyhedra in $\mathbb{R}^3$

Tetrahedron	faces are eq. Δs
Cube	faces are $\square$s
Octahedron	faces are eq. Δs
Dodecahedron	faces are reg. ⬠s
Icosahedron	faces are eq Δs.

All can be triangulated with no extra vertices.



faces of cube



faces of dodecahedron.

The homology groups for these examples are the same $\begin{cases} \mathbb{Z} & \dim 0, 2 \\ 0 & \text{otherwise} \end{cases}$

<u>Also</u> each is homeomorphic to S^2

<u>Discussion</u> All of these are boundaries of <u>convex bodies</u> in $\mathbb{R}^3$.

$K \subseteq \mathbb{R}^n$ compact

convex $\quad x, y \in K \; 0 \le t \le 1 \Rightarrow tx + (1-t)y \in K$

nonempty interior

[regularity] for some continuous $f : \mathbb{R}^n \longrightarrow \mathbb{R}$, $K = f^{-1}[[0,\infty)]$, $\text{bdy}\, K = f^{-1}[\{0\}]$.

Each regular polyhedron bounds a special type of convex body, defined by a finite list of linear inequalities.

(CUBE: $-1 \le x_1, x_2, x_3 \le 1$)

__Theorem__ Every (regular) convex body in $\mathbb{R}^n$ is homeomorphic to D^n such that its (point set) bdy corresponds to S^{n-1}.

The proof is worked out in

convexbodies.pdf.

__Cor.__ Each regular polyhedron is homeomorphic to S^2.

__INVARIANCE QUESTION.__

Given (P, K) and (P', K'). If P is homeo. to P', is

$$H_*(K) \cong H_*(K')? \quad \left[\begin{array}{l}\text{forall } \cancel{\text{any}} \\ \text{vertex} \\ \text{orderings}\end{array}\right].$$

In fact, this is true if P and P' are merely homotopy equivalent.

Several options on how to proceed.

A. Develop algebraic and geometric tools entirely within simplicial homology theory. [original approach]

B. Embed simplicial homology in a more general theory, developing more sophisticated versions of the tools in A.

COMMON DISADVANTAGES OF THESE TWO

1. Developing tools is time consuming.
2. Limited insight into uses of homology.

We shall adopt

C. Set up axioms for embedding simplicial homology in a more general theory, deferring the proof of existence as in "B" to a later time.

This goes back at least to the work of Eilenberg and Steenrod, and these are (greatly) modified versions of their axioms.

DATA

Pairs of topological spaces $A \subseteq X$ and cont maps $f : (X, A) \longrightarrow (Y, B)$ s.t. $f[A] \subseteq B$

Homology groups $H_q(X, A)$, $q \geq 0$, with $H_q = 0$ if $q < 0$.

For each map $f : (X, A) \longrightarrow (Y, B)$,

a homomorphism $f_* : H_*(X, A) \to H_*(Y, B)$

sequence

Natural maps $H_{q+1}(X, A) \xrightarrow{\partial} H_q(A)$

[<u>Note</u> Write $H_q(X, \phi) = H_q(X)$]

For simplicial complexes, natural

SIMPLICIAL SINGULAR

maps $\theta_{(K,P)} : H_*(K, P) \longrightarrow H_*(P)$,

and likewize for pairs, that are isomorphisms and compatible with ∂ maps.

AXIOMS (Functoriality) For maps of pairs $(f_2 f_1)_* = f_{2*} f_{1*}$, $id_* = id$

(Long exact sequences) Natural long exact sequences

$$\cdots \to H_{q+1}(X,A) \to H_q(A) \to H_q(X) \to H_q(X,A) \cdots$$

(Homotopy invariance) If $f \simeq g$ as maps $(X,A) \to (Y,B)$, then $f_* = g_*$.

(Compact supports) If $u \in H_q(X)$, then there is a compact subset $K \subseteq X$ s.t. $u \in \text{Image } H_q(K) \to H_q(X)$; if $u \to 0$ in $H_q(X)$, $u \in H_q(K)$, then there is some compact set $K' \supseteq K$ in X such that $u \to 0$ in $H_q(K')$.

(Excision I) If U is a subset of X such that $\overline{U} \subseteq \text{Interior } A$, $A \subseteq X$, then the inclusion induced mappings $H_*(X-U, A-U) \to H_*(X,A)$ are isomorphisms.

(Excision II) If $X = U \cup V$ where U and V are open in X, then

$$H_*(U, U \cap V) \xrightarrow{\cong} H_*(X, V)$$

(Excision III = Mayer-Vietoris Sequence)

If $X = U \cup V$ where U and V are open in X, then there is a long exact sequence

$$\rightarrow H_{q+1}(X) \xrightarrow{\Delta} H_q(U \cap V) \xrightarrow{(j_{U*}, -j_{V*})} \begin{matrix} H_q(U) \\ \oplus \\ H_q(V) \end{matrix} \xrightarrow{i_{U*} + i_{V*}} H_q(X) \ \vdots$$

which is natural for maps $(X; U, V) \longrightarrow (X'; U', V')$.

Note that the subsets are open, so this differs somewhat from simplicial MV seqs.

(Splitting principle). The inclusions of arc components $X_\alpha \subseteq X$ induce an isomorphism

$$\bigoplus H_*(X_\alpha) \longrightarrow H_*(X)$$

weak direct sum – only finitely many nonzero coordinates.

Furthermore, if X is arcwise connected then $H_0(X) \cong \mathbb{Z}$

(Hurewicz-Poincaré property) If X is arcwise connected, then there is a natural isomorphism

$$\pi_1(X,x)/\text{Commutator subgp} \xrightarrow{\cong} H_1(X,\{x\}).$$

see homology-axioms.pdf for more formal statements.