Consequences of the axioms

Prop. 1 If $f:(X,A)\to(Y,B)$ is a homeomorphism, or even a homotopy equivalence, of pairs, then f_* is an isomorphism. Invariance

Let $g:(Y,B)\to(X,A)$ be a (homotopy) inverse, so $g\circ f\simeq id_{(X,A)}$, $f\circ g\simeq id_{(Y,B)}$. Then $f_*g_*=id_{H_*(Y,B)}$, and $g_*f_*=id_{H_*(X,A)}$.

Prop. 0 $H_*(\phi)=0$ always.

The long exact homology sequence reduces to

$$\rightarrow H_{q+1}(\phi,\phi) \xrightarrow{\partial} H_q(\phi) \xrightarrow{=} H_q(\phi) \xrightarrow{=} H_q(\phi,\phi) \rightarrow$$

$$H_{q+1}(\phi,\phi) = H_{q+1}(\phi)$$

This means $id_* = (id)_* \circ id_* = 0$.

More generally, $H_q(X,X) = 0$ always:

Consider the exact sequence

$$\xrightarrow{\partial} H_q(X) \xrightarrow{=} H_q(X) \xrightarrow{j_*} H_q(X,X) \xrightarrow{\partial} H_{q-1}(X)$$

We know that ∂ must be 0

and j_* must be zero by exactness.

Thus $u \in H_q(X,X)$ & $\partial(u) = 0 \Rightarrow$

$u = j_* v$ some $v \in H_q(X)$; but $j_* = 0$

$\Rightarrow u = j_* V = 0$.

<u>Prop. 2</u> If X is arc wise connected, then the constant map $X \to \{p\}$ defines an isomorphism
$$H_0(X) \xrightarrow{\cong} H_0(\{p\}).$$

<u>Proof</u> $\{p\} \xrightarrow[\text{inclusion of a point}]{} X \xrightarrow[\text{constant}]{} \{p\}$ is the identity, so we have

$$\begin{tikzcd}
H_0(\{pt\}) \arrow[r] & H_0(X) \arrow[r] & H_0(\{pt\}) \\
\mathbb{Z} \arrow[u, phantom, "\shortparallel"] \arrow[r] \arrow[rr, bend right, "\text{identity}"'] & \mathbb{Z} \arrow[u, phantom, "\shortparallel"] \arrow[r] & \mathbb{Z} \arrow[u, phantom, "\shortparallel"]
\end{tikzcd}$$

Therefore $H_0(\{p\}) \to H_0(X)$
$H_0(X) \to H_0(\{p\})$
is an isomorphism.

<u>Cor. 3</u> $H_0(X)$ is free abelian on the set of arc components in X. Also if $f: X \to Y$ maps arc component X_α to Y_β, then f_* sends the corresponding generator $e_\alpha \in H_0(X)$ to $e_\beta \in H_0(Y)$.

Need more concepts to go further, with abstractions, but we can already derive a few significant applications.

Some theorems of L.E.J. Brouwer

[early 20th century]

Thm. S^n is not a retract of D^{n+1}. $n \geq 1$

Proof $i : S^n \xrightarrow{\subseteq} D^{n+1}$ inclusion

If a retraction $r : D^{n+1} \to S^n$ exists,

then $r \circ i = 1_{S^n} \Rightarrow r_* \circ i_* = id_{H_*(S^n)}$

Hence i_* is 1-1. This is not possible

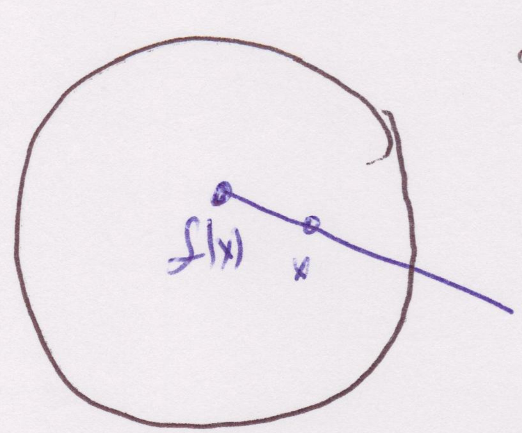
since $H_n(S^n) = \mathbb{Z}$ but $H_n(D^{n+1}) = 0$.

Note If $n = 0$ we would have

$$
\begin{tikzcd}
H_0(S^0) \arrow[r] \arrow[d, equal] & H_0(D^1) \arrow[r] \arrow[d, equal] & H_0(S^0) \arrow[d, equal] \\
\mathbb{Z} \oplus \mathbb{Z} \arrow[r] \arrow[rr, bend right, "\text{identity? NO!!}"'] & \mathbb{Z} \arrow[r] & \mathbb{Z} \oplus \mathbb{Z}
\end{tikzcd}
$$

<u>Brouwer Fixed Point Theorem</u> If $f: D^n \longrightarrow D^n$ $(n \geq 1)$ cont, then there is some $x \in D^n$ s.t. $f(x) = x$.

<u>Proof</u> Assume not. Draw a ray starting at $f(x)$, going through x. It meets S^{n-1} at some point $r(x)$.



<u>CLAIM</u> $r(x)$ cont and $r|S^{n-1} = id$.

For details about r, see brouwer.pdf

<u>Local homology at $x \in X$.</u>

$H_q(X, X - \{x\})$. Assume X is Hausdorff

<u>Lemma</u> $x \in U$ open in $X \Longrightarrow$

$$H_*(U, U-\{x\}) \xrightarrow{\cong} H_*(X, X-\{x\})$$

<u>Proof</u> Apply Excision II to X, U, and $V = X - \{x\}$; note that $U \cap V = U - \{x\}$.

<u>Prop.</u> Suppose $f: X \to Y$ homeo with $f(x) = y$. Then

$$f_*: H_*(X, X-\{x\}) \cong H_*(Y, Y-\{y\}).$$

<u>Brouwer's Invariance of Dimension Thm.</u>

Let U and V be open in $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. If U is homeo to V, then $m = n$.

Find $H_*(U, U-\{x\}) \cong H_*(\mathbb{R}^n, \mathbb{R}^n-\{x\})$ if U is open in $\mathbb{R}^n$.

First note $T(y) = x+y$ induces an iso.

$H_*(\mathbb{R}^n, \mathbb{R}^n-\{0\}) \cong H_*(\mathbb{R}^n, \mathbb{R}^n-\{x\})$.

CLAIM $H_q(\mathbb{R}^n, \mathbb{R}^n-\{0\}) = \begin{cases} \mathbb{Z} & q=n \\ 0 & q\neq n. \end{cases}$

Derivation

① $S^{n-1} \subseteq \mathbb{R}^n-\{0\}$ is a def retract.

② $\rho: \mathbb{R}^n-\{0\} \longrightarrow S^{n-1}$ given by

$\rho(v) = |v|^{-1}v \Rightarrow \rho\circ i = id,\ i\circ\rho \simeq id$

via $H(v,t) = tv + \dfrac{(1-t)}{|v|}v$.

② $Y \subseteq \mathbb{R}^n$ arbitrary $\Longrightarrow$

$\partial: H_{q+1}(\mathbb{R}^n, Y) \xrightarrow{\cong} H_q(Y)$ if $q>0$

$\partial: H_1(\mathbb{R}^n, Y) \cong \text{Kernel}[\text{const. } H_0(Y) \to H_0(pt.)]$

Also, $H_0(\mathbb{R}^n, Y) = 0$.

These isos are natural for inclusions
$Y \subseteq Y' \subseteq \mathbb{R}^n$.

Suppose first that $n \geq 2$, and
$Y = S^{n-1}$ or $\mathbb{R}^n - \{0\}$.

Then $q \geq 2 \Rightarrow H_q(\mathbb{R}^n, Y) \cong H_{q-1}(Y) = \begin{matrix} \mathbb{Z} & q = n \\ 0 & q \neq n \end{matrix}$

$q = 1 \Rightarrow H_1(\mathbb{R}^n, Y) = \text{Kernel } H_0(Y) \longrightarrow H_0(pt.)$

$$\begin{matrix} \| & & \\ \mathbb{Z} & \xrightarrow{\text{onto}} & \mathbb{Z} \end{matrix}$$

$\Rightarrow H_1(\mathbb{R}^n, Y) = 0$

Now suppose $n = 1$. The only nontrivial part of the sequence is

$$\begin{tikzcd}
 & \mathbb{Z}\oplus\mathbb{Z} \arrow[r, "\text{ADD}"] \arrow[d, equal] & \mathbb{Z} \arrow[d, equal] \\
H_1(\mathbb{R}^n, Y) \arrow[r, "1\text{-}1"] & H_0(Y) \arrow[r] & H_0(\mathbb{R}^n) \\
0 = H_1(\mathbb{R}^n) \arrow[u] & &
\end{tikzcd}$$

So that $H_1(\mathbb{R}^n, Y) \cong \mathbb{Z}$.

Proof of Invariance of Dimension

U open in $\mathbb{R}^n \Rightarrow$

$$H_q(U, U - \{u\}) \cong \begin{cases} \mathbb{Z} & q = n \\ 0 & q \neq n \end{cases}$$

V open in $\mathbb{R}^m \Rightarrow$

$$H_q(V, V - \{v\}) \cong \begin{cases} \mathbb{Z} & q = m \\ 0 & q \neq m. \end{cases}$$

These groups are isomorphic $\Longleftrightarrow$

$m = n$.

<u>Complement</u> A similar result holds for topological manifolds.