EXERCISES FOR MATHEMATICS 205C

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DEFAULT HYPOTHESES. Unless specifically stated otherwise, all spaces are assumed to be Hausdorff and locally arcwise connected.

0. (a) Suppose that the 2-dimensional simplicial complex $(P, \mathbf{K})$ has vertices A, B, C, D, E and simplices given by

$$ABC, \quad ABD, \quad ACD, \quad BCD, \quad BCE, \quad BDE, \quad CDE$$

and all their faces. Let $\mathbf{K}_1$ and $\mathbf{K}_2$ be the subcomplexes consisting of all simplices lie in the sets $\{A, B, C, D\}$ and $\{B, C, D, E\}$ respectively. Compute the homology of $(P, \mathbf{K})$ using a Mayer-Vietoris sequence. [*Hint:* Each $\mathbf{K}_i$ is isomorphic to the boundary of a 3-simplex, and their intersection is the simplex BCD.]

(b) Suppose that the 2-dimensional simplicial complex $(P, \mathbf{K})$ has vertices A, B, C, D, X and simplices given by

$$ABC, \quad ABD, \quad ACD, \quad BCD, \quad AX, \quad BX, \quad CX, \quad DX$$

and all their faces. Let $\mathbf{K}_1$ be the subcomplex generated by the first four of these simplices and let $\mathbf{K}_2$ be be the subcomplex generated by the last four. Compute the homology of $(P, \mathbf{K})$ using a Mayer-Vietoris sequence. [*Hint:* Every simplex in the second subcomplex is contained in a simplex whose vertices include X, and the intersection of the two subcomplexes is finite.]

(c) Compute the homology of $(P, \mathbf{K})$ if $\mathbf{K}$ is the 2-skeleton of the simplex Δ_4; recall that the m-skeleton of a simplicial complex is generated by all simplices of dimension $\leq m$. [*Hint:* Computing the relative homology of $(\Delta_4, \mathbf{K})$ is fairly easy to do; note that it is nonzero in only two dimensions.]

1. Prove that the homotopy invariance axiom for homology theories is equivalent to the following weaker statement, which is supposed to hold for all pairs (X, A):

> If $t = 0, 1$ and $i_t : (X, A) \to (X \times [0, 1], A \times [0, 1])$ is the slice inclusion $i_t(x) = (x, t)$, then the maps in homology i_{0*} and i_{1*} are equal.

[*Hint:* If H is a homotopy from f_0 to f_1 then $f_t = H \circ i_t$ for $t = 0, 1$.]

2. Given a topological space X, the **cone** on X, written $\mathbf{C}(X)$, is defined to be the quotient space of $X \times [0, 1]$ whose equivalence classes are the one point subsets (x, t) for $t > 0$ and the subset $X \times \{0\}$, which is called the *vertex* of the cone.

(a) If $f : X \to Y$ is a continuous mapping, explain why the mapping $f \times \mathrm{id}_{[0,1]}$ from $X \times [0,1]$ to $Y \times [0,1]$ passes to a continuous map $\mathbf{C}(f) : \mathbf{C}(X) \to \mathbf{C}(Y)$ and this construction has the functorial properties $\mathbf{C}(g \circ f) = \mathbf{C}(g) \circ \mathbf{C}(f)$ and $\mathbf{C}(\mathrm{id}) = \mathrm{id}$.

(b) Prove that $\mathbf{C}(X)$ is contractible [*Hint:* Show that the identity map is homotopic to the map which sends everything to the vertex.] Using this, show that $H_q(\mathbf{C}(X), X \times \{1\})$ is isomorphic to $H_{q-1}(X)$ if $q \geq 2$, and $H_1(\mathbf{C}(X), X \times \{1\}) = 0$ if X is arcwise connected.

3. Given a topological space X, the **suspension** on X, written $\Sigma(X)$, is defined to be the quotient space of $X \times [-1,1]$ whose equivalence classes are the one point subsets (x,t) for $t \neq \pm 1$ and the subsets $X \times \{-1\}$ and $X \times \{1\}$, which are called the *poles* of the cone (sometimes the latter are called the south and north poles respectively and written P_- and P_+).

(a) If $f : X \to Y$ is a continuous mapping, explain why the mapping $f \times \mathrm{id}_{[-1,1]}$ from $X \times [-1,1]$ to $Y \times [-1,1]$ passes to a continuous map $\Sigma(f) : \Sigma(X) \to \Sigma(Y)$ and this construction has the functorial properties $\Sigma(g \circ f) = \Sigma(g) \circ \Sigma(f)$ and $\Sigma(\mathrm{id}) = \mathrm{id}$.

(b) Prove that $\Sigma(X) - \{P_-\}$ and $\Sigma(X) - \{P_+\}$ are contractible, and in fact they are deformation retracts of $\{P_+\}$ and $\{P_-\}$ respectively. Also explain why $\Sigma(X) - \{P_+, P_-\}$ is homotopy equivalent to X. [*Hint:* Why is it homeomorphic to $X \times (-1,1)$?]

(c) Prove that if X is arcwise connected then $H_1\big(\Sigma(X)\big) = 0$, while if $q \geq 2$ then there is an isomorphism from $H_q\big(\Sigma(X)\big)$ to $H_{q-1}(X)$ which is essentially given by the boundary map in a suitable Mayer-Vietoris exact sequence. [*Hint:* Use the decomposition

$$\Sigma(X) \quad = \quad \Sigma(X) - \{P_-\} \ \cup \ \Sigma(X) - \{P_+\}$$

and the conclusions of part (b) above. What is the intersection of the two given open subsets?]

3. Suppose that M_1 and M_2 are connected topological n-manifolds with $p_i \in M_1$, and let $M_1 \vee M_2$ denote the quotient of the disjoint union $M_1 \amalg M_2$ in which p_1 and p_2 are identified. — Informally, this space is a one point union of M_1 and M_2 which contains (homeomorphic copies of) the latter as closed subspaces, and the intersection of these subspaces is a single point.

(a) Explain why $M_i \subset M_1 \vee M_2$ has an open neighborhood U_i such that M_i is a deformation retract of U_i. [*Hint:* Consider the subset $M_1 \vee W_2$ where W_2 is a neighborhood of p_2 which is homeomorphic to an open n-disk, and do the same with the roles of 1 and 2 reversed.]

(b) Prove that if $q > 0$ then $H_q(M_1 \vee M_2)$ is isomorphic to $H_q(M_1) \oplus H_q(M_2)$.

4. (a) If X is a nonempty space prove that $H_q(S^1 \times X)$ is isomorphic to $H_q(X) \oplus H_{q-1}(X)$. [*Hint:* Write $S^1 = U \cup V$ where $U - S^1 = \{1\}$ and $V = S^1 - \{-1\}$, consider the corresponding decomposition of $S^1 \times X$, and look at the associated Mayer-Vietoris exact sequence.]

(b) Using the same ideas and an induction argument, prove that $H_q(S^n \times X)$ is isomorphic to $H_q(X) \oplus H_{q-n}(X)$. [*Hint:* Write $S^n = U \cup V$ where U and V are complements of the "north and south poles" $\pm \mathbf{e}_{n+1}$, and observe that $U \cap V \cong S^{n-1} \times \mathbb{R}$.]

5. (a) Explain why $H_0(X, A)$ is a free abelian group on the set of all arc components of X which do not contain any points of A.

(b) Compute the homology of (S^n, A) where $n \geq 1$ and A is a finite set. There are two cases depending upon whether or not $n = 1$.

6. (*a*) Prove that the sphere S^m is not a retract of the sphere S^n if $m \neq m$.

(*b*) Suppose that $A \subset X$ is a retract of X. Prove that $H_q(X) \cong H_q(A) \oplus H_q(X, A)$ for all $q \geq 0$.

7. Let $f : (D^n, S^{n-1}) \to (D^n, D^n - \{0\})$ be the inclusion map of pairs. Show that f defines homotopy equivalences from D^n to itself and from S^{n-1} to $D^n - \{0\}$, but f is not a homotopy equivalence of pairs. [*Hint:* Note that a homotopy equivalence of Hausdorff pairs $(X, A) \to (Y, B)$ also defines a homotopy equivalence of pairs $(X, \overline{A}) \to (Y, \overline{B})$, where as usual $\overline{C}$ denotes the closure of the subspace C.]

8. If $d > 1$ is an integer, then the *generalized Möbius strip* M_d is defined to be the quotient of $S^1 \times [0, 1]$ whose equivalence classes are one point sets corresponding to the points (z, t) where $t < 1$ and subsets with d elements of the form $\{(\alpha z, t) \mid \alpha^d = 1\}$. If $d = 2$ this is just the usual Möbius strip.

(*a*) Let $B \subset M_d$ denote the image of $S^1 \times \{1\}$. Prove that B is a deformation retract of M_d. [*Hint:* Show that the deformation retraction ρ of $S^1 \times [0, 1]$ which pushes everything down vertically to $S^1 \times \{1\}$ passes to a map $M_d \to B$ and that the homotopy from the identity to the composite of ρ with the inclusion $S^1 \times \{1\} \subset S^1 \times [0, 1]$ passes to a homotopy of the composite $M_d \to B \subset M_d$.

(*b*) Let $q : S^1 \cong S^1 \times \{1\} \to B \subset M_d$ be induced by the quotient space projection from $S^1 \times [0, 1]$ to M_d. Prove that there is a homeomorphism h from B to S^1 such that the composite $h \circ q$ is the map from S^1 to itself which sends z to z^d. Also, explain why q is homotopic to the composite

$$S^1 \approx S^1 \times \{0\} \subset S^1 \times [0, 1] \to M_d$$

where the morphism on the right is just the quotient projection. [*Hint for the first part:* Show first that there is a continuous map $h : B \to S^1$ such that $h \circ q(z) = z^d$ by noting that if two points in S^1 map to the same equivalence class in B then their images under the map $z \to z^d$ are equal. Then verify that h is 1–1 onto and hence is a homeomorphism onto its image. — The second statement will follow easily from the fact that the slice inclusions i_0, i_1 of S^1 in $S^1 \times [0, 1]$ are homotopic.]

(*c*) Let $A_k \subset S^1$ be the minor arc whose endpoints are β^{k-1} and β^k, where $\beta = \exp(2\pi\, i/3d)$. Explain why $A_k \times [0, 1]$ maps homeomorphically onto its image in M_d, and using the standard homeomorphisms $A_k \times [0, 1] \cong [0, 1] \times [0, 1]$ construct a triangulation of M_d whose vertices correspond to the points $(\beta^k, 0)$ and $[\beta^k, 1]$, where "[-,-]" denotes the associated equivalence class in M_d. Note that there are $3d$ vertices of the first type and 3 of the second. [*Hint:* The 2-simplices should have vertices corresponding to triples of the forms $\{(\beta^k, 0), (\beta^{k+1}, 0), [\beta^k, 1]\}$ and $\{(\beta^k, 0), [\beta^k, 1], [\beta^{k+1}, 1]\}$. Thus there are $6d$ simplices of dimension 2, with $3d + 3$ vertices and $9d + 3$ edges. There is a drawing for this problem in the file `exercises04a.pdf` .]

(*d*) Let $\mathbf{K}$ be the simplicial complex obtained in the preceding discussion, let $\mathbf{L}$ be the subcomplex corresponding to $S^1 \times \{0\}$, and attach a cone $\mathbf{Q}$ on $\mathbf{L}$ whose simplices have the form wvv' and their boundaries, where vv' is an edge of $\mathbf{L}$. Let $\mathbf{N}_d$ denote the union of $\mathbf{K}$ and $\mathbf{Q}$ along $\mathbf{L}$, and let $P[d]$ denote the underlying space. Prove that $H_q(P[d])$ is isomorphic to $\mathbb{Z}$ if $q = 0$, $\mathbb{Z}_d$ if $q = 1$, and 0 otherwise. [*Hint:* Use a Mayer-Vietoris sequence for simplicial homology together with the provious observations that $q_* : H_1(S^1) \to H_1(B)$ is multiplication by d, the inclusion map defines a homotopy equivalence from B to M_d, and q is homotopic to the composite $S^1 \approx S^1 \times \{0\} \to S^1 \times [0, 1] \to M_d$.] that is multiplication by d on H_1 of the appropriate spaces under the given isomorphism H

9. For $1 \leq q \leq n$ let G_n be a finitely generated abelian group. Prove that there is a connected polyhedron P such that $H_q(P) = G_q$ for $1 \leq q \leq n$. [*Hints:* Use 8(d) and 3(b) to do the case where $n = 1$, and use 8(d), 3(b) and suspensions to get the conclusion in higher dimensions.]

10. (a) Using the Mayer-Vietoris sequence for the decomposition of $T^2 - \{(-1,-1)\}$ into $T^2 - (S^1 - \{-1\}) \cup (S^1 - \{-1\}) \times S^1$, prove that H_1 of this arcwise connected space is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$ in dimension 1 and is zero in all other positive dimensions. [*Hint:* S^1 minus a point is contractible, as is the intersection of the open sets in the decomposition.]

(b) Using Exercise 4, compute the homology of $T^2 = S^1 \times S^1$ and show that projection onto either coordinate induces a surjection from $H_1(T^2)$ to $H_1(S^1)$. [*Hint:* Explain why it suffices to consider projection onto the second coordinate and use some steps in the proof of Exercises 4.]

(c) Using the preceding observations, explain why the inclusion of $T^2 - \{(-1,-1)\}$ in T^2 induces an isomorphism in homology. [*Hint:* Let j_1, j_2 denote inclusions of S^1 as $S^1 \times \{1\}$ and $\{1\} \times S^1$ respectively, and let π_1, π_2 denote the coordinate projections. Explain why $\pi_{s*} \circ \pi_{t*}$ is the identity if $s = t$ and trivial if $s \neq t$.]

(d) Let W be an open neighborhood of $(-1,-1)$ in T^2 such that W is homeomorphic to an open 2-disk with $(-1,1)$ corresponding to its center. By excision we know that $H_*(W, W - \{(-1,-1)\}) \cong H_*(T^2, T^2 - \{(-1,-1)\})$. Prove that the 2-dimensional generator of this group maps to zero in $H_1(T^2 - \{(-1,-1)\})$ and hence lies in the image of $H_2(T^2)$. Using this show that the inclusion of $W - \{(-1,-1)\} \cong S^1 \times (0,1)$ in $T^2 - \{(-1,-1)\}$ induces the zero map in homology.

(e) The double torus or oriented surface of genus two has a decomposition of the form $U_1 \cup U_2$ where U_i is homeomorphic to $T^2 - \{(-1,-1)\}$ and the intersection is given by $W_1 - \{(-1,-1)\} \subset U_1$ or equivalently $W_2 - \{(-1,-1)\} \subset U_2$. Compute the homology groups of the double torus using this decomposition and a Mayer-Vietoris sequence.

11. Suppose that U and V are open convex subsets of $\mathbb{R}^n$ and $U \cap V$ is nonempty. Prove that $H_q(U \cup V) = 0$ if $q \neq 0$ and $H_0(U \cup V) \cong \mathbb{Z}$, and give an example to show that $U \cup V$ is not necessarily convex. [*Hint:* $U \cap V$ is convex and if C is convex then C is contractible.]

12. ($\star$) Suppose we are given a triple of spaces (X, A, B) such that A is a subspace of X and B is a subset of A. For each integer q, define the connecting homomorphism ∂_* from $H_{q+1}(X, A)$ to $H_q(A, B)$ to be the composite $H_{q+1}(X, A) \to H_q(A) \to H_q(A, B)$. Prove that the sequence

$$\cdots \to H_{q+1}(X, A) \to H_q(A, B) \to H_q(X, B) \to H_q(X, A) \to H_{q-1}(B, A) \cdots$$

is exact. [One reference for a proof is the book by Eilenberg and Steenrod.]