REDUCED ~~RELATIVE~~ MAYER-VIETORIS SEQUENCES

If $X \neq \phi$ then there is a unique continuous map $c_X : X \longrightarrow \{0\}$. This map is a retraction — we can let $\rho(0) = p$ where $p \in X$ is arbitrary.

<u>LEMMA.</u> $H_q(X) \cong H_q(pt.) \oplus \text{Kernel}\, c_{X*}$.

<u>SKETCH OF PROOF.</u> Let ρ be as above. Then $c_{X*} \circ \rho_* = id_{H_q(X)} \Longrightarrow H_q(X) \cong \text{Image}\, \rho_* \oplus \text{Kernel}\, c_{X*}$. But ρ_* is 1-1 $(c_* \rho_* = id)$, so $\text{Image}\, \rho_* \cong H_q(pt.)$. ■

<u>Def.</u> $\text{Kernel}\, c_{X*} = \tilde{H}_q(X)$, <u>reduced homology</u>.

<u>Prop.</u> $\tilde{H}_*(X)$ is a subfunctor of $H_*(X)$.

<u>Proof</u> We have a commutative diagram

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "c_X"'] & & Y \arrow[dl, "c_Y"] \\ & & \end{tikzcd}$$

so that $u \in \text{Ker}\, c_{X*} \Longrightarrow f_*(u) \in \text{Ker}\, c_{Y*}$. ■

<u>Note</u> $\tilde{H}_q = H_q$ if $q > 0$, $\tilde{H}_0 \oplus \mathbb{Z} \cong H_0$ naturally.

<u>Slogan</u> $\tilde{H}_q$ is obtained by removing the trivial part of H_q. If X is nonempty we know that $H_0(X)$ is free abelian on at least one generator, and $\tilde{H}_0(X)$ describes all the other generators (but there may be none!).

<u>Theorem</u> Let $X = U \cup V$ where U, V are open in X and $U \cap V \neq \phi$ ($\Rightarrow$ same for U, V, X). Then one has a relative Mayer-Vietoris exact sequence and a canonical map from it to the ordinary M-V exact sequence:

$$\begin{array}{ccccccccc}
\to \tilde{H}_{q+1}(X) & \to & \tilde{H}_q(U \cap V) & \to & \tilde{H}_q(U) \oplus \tilde{H}_q(V) & \to & \tilde{H}_q(X) & \to & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\to H_{q+1}(X) & \to & H_q(U \cap V) & \to & H_q(U) \oplus H_q(V) & \to & H_q(X) & \to & \cdots
\end{array}$$

<u>Proof</u> Since $\tilde{H}_q \to H_q$ is an isomorphism if $q \neq 0$, it is only necessary to check what happens when $q = 0$. In this case we claim we have the following large diagram, where the map from

the second row to the third is induced by the constant map of triads $(X; U, V) \to (\{0\}; \{0\}, \{0\})$:

$$\begin{tikzcd}
\arrow[r] & \tilde{H}_1(X) \arrow[r, "\Delta"] \arrow[d, "\cong"] \arrow[dr, phantom, "\circledast"] & \tilde{H}_0(U \cap V) \arrow[r, "{(i_{U*}, -i_{V*})}"] \arrow[d] \arrow[dr, phantom, "?"] & \tilde{H}_0(U) \oplus \tilde{H}_0(V) \arrow[r, "j_{U*} + j_{V*}"] \arrow[d] \arrow[dr, phantom, "??"] & \tilde{H}_0(X) \arrow[r] \arrow[d] & 0 \arrow[d, "="] \\
\arrow[r] & H_1(X) \arrow[r] \arrow[d] & H_0(U \cap V) \arrow[r] \arrow[d] & H_0(U) \oplus H_0(V) \arrow[r] \arrow[d] & H_0(X) \arrow[r] \arrow[d] & 0 \arrow[d, "="] \\
\arrow[r] & H_1(\{0\}) \arrow[r] \arrow[d, leftrightarrow] & H_0(\{0\}) \arrow[r] \arrow[d, leftrightarrow] & H_0\{0\} \oplus H_0(\{0\}) \arrow[r] \arrow[d] & H_0\{0\} \arrow[r] \arrow[d] & 0 \\
& 0 \arrow[r] & \mathbb{Z} \arrow[r] & \mathbb{Z} \oplus \mathbb{Z} \arrow[r, "\text{ADD}"] & \mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

$$n \longrightarrow (n, -n)$$

<u>Note</u>: We had not previously verified the commutativity of $\circledast$, but it follows quickly from the commutativity of the square directly below it.

Similarly, we need to verify that the squares ? and ?? exist. If so, then exactness of the top line will follow from exactness of the second line (note that the vertical arrows 1st $\rightsquigarrow$ 2nd are all 1-1 mappings!)

Commutativity of [?]

If $y \in H_0(U \cap V)$ and $c_*(y) = 0$, consider $(i_{U*}y, -i_{V*}y) \in H_0(U) \oplus H_0(V)$. We need to show that $c_* \oplus c_*$ maps this to zero. But $c_* \oplus c_*$ on the given element is $(c_* i_{U*} y, -c_* i_{V*} y)$ $= (c_* y, -c_* y) = (0, 0)$, so $(i_{U*}, -i_{V*})$ maps $\tilde{H}_0(U \cap V)$ into $\tilde{H}_0(U) \oplus \tilde{H}_0(V)$.

Commutativity of [??]

Same idea. If (y, z) satisfies $c_* y = 0$, $c_* z = 0$, then

$$c_*\big(j_{U*}(y) + j_{V*}(z)\big) = c_* j_{U*}(y) + c_* j_{V*}(z) =$$

$$c_*(y) + c_*(z) = 0 + 0 = 0.$$

Verification that the top line is exact

Since the objects in the first line are subgroups of the corresponding objects in the second, it follows that all ~~2~~ 2-fold composites $\cdot \longrightarrow \cdot \longrightarrow \cdot$

in the first line are zero, which yields half of the exactness conditions. The remaining parts only require a little more work.

<u>Exactness at $\tilde{H}_1(X)$.</u> If $u \in \tilde{H}_1(X)$ and $\Delta(u)=0$, by exactness of the ordinary MV sequence we have $u = j_{U*}(y) + j_{V*}(z)$ for suitable y, z.

<u>Exactness at $\tilde{H}_0(U \cap V)$</u> If $u \in \tilde{H}_0(U\cap V)$ and $i_{U*}(u)$, $i_{V*}(u) = 0$, then $u = \Delta w$ for some w by the usual MV exact sequence.

<u>Exactness at $\tilde{H}_0(U) \oplus \tilde{H}_0(V)$</u> If $j_{U*}(y) + j_{V*}(z) = 0$ for $(y,z) \in \tilde{H}_0(U) \oplus \tilde{H}_0(V)$, then inside $H_0(U) \oplus H_0(V)$ we have $(y,z) = (i_{U*}a, -i_{V*}a)$ for some $a \in H_0(U\cap V)$. We need to check that $c_*(a) = 0$, so that $a \in \tilde{H}_0(U \cap V)$. But $c_*(a) = c_* i_{U*}(a) = c_*(y) = 0$ because $y \in \tilde{H}_0(U)$.

Exactness at $\tilde{H}_0(X)$. Suppose that $\boxed{w \in \tilde{H}_0(X).}$ hence $c_*(w)=0$ Then by the usual exact MV sequence we have $w = j_{U*}y + j_{V*}z$ for some $(y,z) \in H_0(U) \oplus H_0(V)$, and all we need to do is show that we can take y and z to lie in the reduced groups.

Unfortunately, the condition $c_*(w)=0$ only implies $c_* j_{U*} y + c_* j_{V*} z = 0$

Let $k \in H_0(\{0\})$ be such that $k = c_* j_{U*} y = - c_* j_{V*} z$, and let $p \in U \cap V$ with $\rho_U, \rho_V : \{0\} \longrightarrow U, V$ mapping 0 to p. Modify y, z by the formulas

$$\begin{cases} \bar{y} = y - \rho_{U*}(k) \\ \bar{z} = z + \rho_{V*}(k). \end{cases}$$

Then $c_*(j_{U*}(\bar{y})) = 0 = c_*(j_{V*}(\bar{z}))$, so $\bar{y} \in \tilde{H}_0(U)$, $\bar{z} \in \tilde{H}_0(V)$, and if $p : \{0\} \longrightarrow X$ sends 0 to p we have

~~$j_{U*}(\bar{y}) + j_{V*}(\bar{z}) = j_{U*}(y + \rho(k)) + j_{V*}(z + \rho(k))$~~

$$j_{U*}(\bar{y}) + j_{V*}(\bar{z}) = j_{U*}(y - \rho_*(k)) + j_{V*}(z + \rho_*(k)) =$$

$$j_{U*}(y) + j_{V*}(z) + \underbrace{j_{U*}\rho_{U*}(-k) + j_{V*}\rho_{V*}(k)}_{= -\rho_*(k) \qquad = +\rho_*(k)}.$$

so these cancel

Since the remaining expression is just w, we have shown $w = j_{U*}(\bar{y}) + j_{V*}(\bar{z})$ where $(\bar{y}, \bar{z}) \in \tilde{H}_0(U) \oplus \tilde{H}_0(V)$. This completes the proof of exactness. ∎

<u>COMPLEMENT.</u> The reduced MV sequence is natural for maps of triads

$$(X; U, V) \longrightarrow (X'; U', V')$$

where $U \cap V, U' \cap V' \neq \phi$ and U, V are open in X and U', V' are open in X'. ∎