Reduced versus relative homology

RECALL $X \neq \phi$ and $c_X : X \longrightarrow \{pt\}$.
constant map

$\Longrightarrow \tilde{H}_q(X) \cong \text{Kernel } c_{X*}$.
reduced

CLAIM If $x_0 \in X$, then
$$\tilde{H}_q(X) \cong H_q(X, x_0).$$

$c_X \circ i = 1_{\{x_0\}}$

Derivation: Consider
$$\{x_0\} \xrightarrow[i]{\subseteq} X \xrightarrow[\overset{\shortparallel}{c_X}]{\text{const}} \{x_0\}$$ which yields

an isomorphism $H_q(X) \cong \tilde{H}_q(X) \oplus c_{X*}[H_q(\{x_0\})]$

Now look at the long exact homology sequence for $(X, \{x_0\})$. Since i is a retract and $c_X \circ i = \text{identity}$, we obtain a splitting
$$H_q(X) \cong c_{X*}[H_q(\{x_0\})] \oplus H_q(X, x_0).$$
(SEE THE EXERCISES!!)

Therefore both $H_q(X, x_0)$ and $\tilde{H}_q(X)$ are given by

$$H_q(X) / \text{Image } c_{X*}$$

and hence they must be isomorphic. ∎