III.4 : Amalgamation theorems

(Conlon, § 1.3)

Many physical and geometrical objects are describable in terms of pieces that are somehow glued together. This principle applies particularly to the theory of smooth manifolds. The purpose of this section is to develop the mathematical concepts and results that are needed to assemble topological spaces and smooth manifolds from a collection of pieces. More formally, we must answer the following question at the topological level: Given a collection of topological spaces, what sorts of data do we need in order to glue them together and form a single "reasonable" space? Most of the time we also want a simple additional condition; namely, the space we construct should have an open, or finite closed, covering consisting of subsets homeomorphic to the objects in the original collection.

Since it is generally useful to have specific examples when setting up abstract mathematical machinery, here is one that is fairly simple and familiar but not entirely trivial: Physically it is clear that one can form a cube from six pairwise disjoint squares with sides of equal length by gluing the latter together in a suitable way along the edges. Whatever formalism we develop should provide a mathematical model for this well known process.

III.4.A. : Topological amalgamation

Since smooth manifolds are topological spaces with additional structure, we shall begin by discussing the underlying topological concepts and results. The first step is to introduce some constructions that are elementary and necessary for this course but do not appear in most point set topology texts (including [MUNKRES1]!).

III.4.A.1 : *Disjoint unions*

We shall need an elementary set-theoretic construction that is described in the ONLINE 205A NOTES. Namely, given two sets A and B we need to have a *disjoint union*, written $A \sqcup B$ or $A \coprod B$, which is a union of two disjoint subsets that are essentially xerox copies of A and B.

Most texts and courses on set theory and point set topology (*e.g.*, [MUNKRES1]) do not say much if anything about disjoint union constructions, one reason being that everything is fairly elementary when one finally has the right definitions (two references in print are Section 8.7 of Royden, *Real Analysis*, and Sections I.3 and III.4–III.7 of the text by K. Jänich mentioned at the beginning of these notes).

Since constructions of this sort play a crucial role beginning with the next section of these notes, a brief but comprehensive treatment seems worthwhile for the sake of precision and clarity.

Formally, **the disjoint union** (or **set-theoretic sum**) of two sets A and B is defined to be the set

$$A \coprod B \quad = \quad \big(A \times \{1\}\big) \bigcup \big(B \times \{2\}\big) \quad \subset \quad \big(A \cup B\big) \times \{1, 2\}$$

with **injection** maps $i_A : A \to A \coprod B$ and $i_B : A \to A \coprod B$ given by $i_A(a) = (a, 1)$ and $i_B(b) = (b, 2)$. The images of these injections are disjoint copies of A and B, and the union of the images is $A \coprod B$.

Definition. If X and Y are topological spaces, the *disjoint union topology* or *(set-theoretic) sum topology* on the set $X \coprod Y$ consists of all subsets having the form $U \coprod V$, where U is open in X and V is open in Y.

We claim that this construction defines a topology on $X \coprod Y$, and the latter is a union of disjoint homeomorphic copies of X and Y such that each of the copies is an open and closed subset. Formally, all this is expressed as follows:

ELEMENTARY PROPERTIES. *The family of subsets described above is a topology for $X \coprod Y$ such that the injection maps i_X and i_Y are homeomorphisms onto their respective images. These images are pairwise disjoint, and they are also open and closed subspaces of $X \coprod Y$. Each injection map is continuous, open and closed.*

Sketch of proof. This is all pretty elementary, but we include it because the properties are so fundamental and the details are not readily available in the standard texts.

Since X and Y are open in themselves and $\emptyset$ is open in both, it follows that $X \coprod Y$ and $\emptyset = \emptyset \coprod \emptyset$ are open in $X \coprod Y$. Given a family of subsets $\{ U_\alpha \coprod V_\alpha \}$ in the so-called disjoint union topology, then the identity

$$\bigcup_\alpha \left(U_\alpha \coprod V_\alpha \right) \quad = \quad \left(\bigcup_\alpha U_\alpha \right) \coprod \left(\bigcup_\alpha V_\alpha \right)$$

shows that the so-called disjoint union topology is indeed closed under unions, and similarly the if $U_1 \coprod V_1$ and $U_2 \coprod V_2$ belong to the so-called disjoint union topology, then the identity

$$\bigcap_{i=1,2} \left(U_i \coprod V_i \right) \quad = \quad \left(\bigcap_{i=1,2} U_i \right) \coprod \left(\bigcap_{i=1,2} V_i \right)$$

shows that the so-called disjoint union topology is also closed under finite intersections. In particular, we are justified in calling this family a topology.

By construction U is open in X if and only if $i_X(U)$ is open in $i_X(X)$, and V is open in Y if and only if $i_Y(V)$ is open in $i_Y(Y)$; these prove the assertions that i_X and i_Y are homeomorphisms onto their images. Since $i_X(X) = X \coprod \emptyset$, it follows that the image of i_X is open, and of course similar considerations apply to the image of i_Y. Also, the identity

$$i_X(X) = \left(X \coprod Y \right) \; - \; i_Y(Y)$$

shows that the image of i_X is closed, and similar considerations apply to the image of i_Y.

The continuity of i_X follows because every open set in $X \coprod Y$ has the form $U \coprod V$ where U and V are open in X and Y respectively and

$$i_X^{-1} \left(U \coprod V \right) \quad = \quad U$$

with similar conditions valid for i_Y. The openness of i_X follows immediately from the identity $i_X(U) = U \coprod \emptyset$ and again similar considerations apply to i_Y. Finally, to prove that i_X is closed, let $F \subset X$ be closed. Then $X - F$ is open in X and the identity

$$i_X(F) = F \coprod \emptyset = \left(X \coprod Y\right) - \left((X - F) \coprod Y\right)$$

shows that $i_X(F)$ is closed in $X \coprod Y$; once more, similar considerations apply to i_Y.■

IMMEDIATE CONSEQUENCE. *The closed subsets of $X \coprod Y$ with the disjoint union topology are the sets of the form $E \coprod F$ where E and F are closed in X and Y respectively.*■

If the topologies on X and Y are clear from the context, we shall generally assume that the $X \coprod Y$ is furnished with the disjoint union topology unless there is an explicit statement to the contrary.

Since the disjoint union topology is not covered in many texts, we shall go into more detail than usual in describing their elementary properties.

FURTHER ELEMENTARY PROPERTIES. (*i*) *If X and Y are discrete, then so is $X \coprod Y$.*

(*ii*) *If X and Y are Hausdorff, then so is $X \coprod Y$.*

(*iii*) *If X and Y are homeomorphic to metric spaces, then so is $X \coprod Y$.*

(*iv*) *If $f : X \to W$ and $g : Y \to W$ are continuous maps into some space W, then there is a unique continuous map $h : X \coprod Y \to W$ such that $h \circ i_X = f$ and $h \circ i_Y = g$.*

(*v*) *The spaces $X \coprod Y$ and $Y \coprod X$ are homeomorphic for all X and Y. Furthermore, if Z is a third topological space then there is an "associativity" homeomorphism*

$$\left(X \coprod Y\right) \coprod Z \quad \cong \quad X \coprod \left(Y \coprod Z\right)$$

(in other words, the disjoint sum construction is commutative and associative up to homeomorphism).

Sketches of proofs. (*i*) A space is discrete if every subset is open. Suppose that $E \subset X \coprod Y$. Then E may be written as $A \coprod B$ where $A \subset X$ and $B \subset Y$. Since X and Y are discrete it follows that A and B are open in X and Y respectively, and therefore $E = A \coprod B$ is open in $X \coprod Y$. Since E was arbitrary, this means that the disjoint union is discrete.

(*ii*) If one of the points p, q lies in the image of X and the other lies in the image of Y, then the images of X and Y are disjoint open subsets containing p and q respectively. On the other hand, if both lie in either X or Y, let V and W be disjoint open subsets containing the preimages of p and q in X or Y. Then the images of V and W in $X \coprod Y$ are disjoint open subsets that contain p and q respectively.

(*iii*) As noted in Theorem 20.1 on page 121 of [MUNKRES1], if the topologies on X and Y come from metrics, one can choose the metrics so that the distances between two points are ≤ 1. Let $\mathbf{d}_X$ and $\mathbf{d}_Y$ be metrics of this type.

Define a metric $\mathbf{d}^*$ on $X \coprod Y$ by $\mathbf{d}_X$ or $\mathbf{d}_Y$ for ordered pairs of points (p, q) such that both lie in the image of i_X or i_Y respectively, and set $\mathbf{d}^*(p, q) = 2$ if one of p, q lies in the image of i_X and the other lines in the image of i_Y. It follows immediately that $\mathbf{d}^*$ is nonnegative, is zero if and only if $p = q$ and is symmetric in p and q. All that remains to check is the Triangle Inequality:

$$\mathbf{d}^*(p, r) \quad \leq \quad \mathbf{d}^*(p, q) + \mathbf{d}^*(q, r)$$

The verification breaks down into cases depending upon which points lie in the image of one injection and which lie in the image of another. If all three of p, q, r lie in the image of one of the injection maps, then the Triangle Inequality for these three points is an immediate consequence of the corresponding properties for $\mathbf{d}_X$ and $\mathbf{d}_Y$. Suppose now that p and r lie in the image of one injection and q lies in the image of the other. Then we have $\mathbf{d}^*(p,r) \leq 1$ and

$$\mathbf{d}^*(p,q) \;+\; \mathbf{d}^*(q,r) \quad = \quad 2+2 \quad = \quad 4$$

so the Triangle Inequality holds in these cases too. Finally, if p and r lie in the images of different injections, then either p and q lie in the images of different injections or else q and r lie in the images of different injections. This means that $\mathbf{d}^*(p,r) = 2$ and $\mathbf{d}^*(p,q) + \mathbf{d}^*(q,r) \geq 2$, and consequently the Triangle Inequality holds for **all** ordered pairs (p,r).

(iv) Define $h(x,1) = f(x)$ and $h(y,2) = g(y)$ for all $x \in X$ and $y \in Y$. By construction $h \circ i_X = f$ and $h \circ i_Y = g$, so it remains to show that h is continuous and there is no other continuous map satisfying the functional equations. The latter is true for set theoretic reasons; the equations specify the behavior of h on the union of the images of the injections, but this image is the entire disjoint union. To see that h is continuous, let U be an open subset of X, and consider the inverse image $U^* = h^{-1}(U)$ in $X \coprod Y$. This subset has the form $U^* = V \coprod W$ for some subsets $V \subset X$ and $W \subset Y$. But by construction we have

$$V \quad = \quad i_X^{-1}(U^*) \quad = \quad i_X^{-1} \circ h^{-1}(U) \quad = \quad f^{-1}(U)$$

and the set on the right is open because f is continuous. Similarly,

$$W \quad = \quad i_Y^{-1}(U^*) \quad = \quad i_Y^{-1} \circ h^{-1}(U) \quad = \quad g^{-1}(U)$$

so that the set on the right is also open. Therefore $U^* = V \coprod W$ where V and W are open in X and Y respectively, and therefore U^* is open in $X \coprod Y$, which is exactly what we needed to prove the continuity of h.

(v) We shall merely indicate the main steps in proving these assertions and leave the details to the reader as an exercise. The homeomorphism τ from $X \coprod Y$ to $Y \coprod X$ is given by sending $(x,1)$ to $(x,2)$ and $(y,2)$ to $(y,1)$; one needs to check this map is 1–1, onto, continuous and open (in fact, if τ_{XY} is the map described above, then its inverse is τ_{YX}). The "associativity homeomorphism" sends $(\,(x,1),1)$ to $(x,1)$, $(\,(y,2),1)$ to $(\,(y,1),2)$, and $(z,2)$ to $(\,(z,2),2)$. Once again, one needs to check this map is 1–1, onto, continuous and open.∎

COMPLEMENT. *There is an analog of Property* (iv) *for untopologized sets.*

Perhaps the fastest way to see this is to make the sets into topological spaces with the discrete topologies and then to apply (i) and (iv).∎

Property (iv) is dual to the fundamental defining property of direct products. Specifically, ordered pairs of maps from a fixed object A to objects B and C correspond to maps from A into $B \times C$, while ordered pairs of maps going **TO** a fixed object A and coming **FROM** objects B and C correspond to maps from $B \coprod C$ into A. For this reason one often refers to $B \coprod C$ as the **coproduct** of B and C (either as sets or as topological spaces); this is also the reason for denoting disjoint unions by the symbol $\coprod$, which is merely the product symbol $\prod$ turned upside down.

III.4.A.2 : *Copy, cut and paste constructions* ($1\frac{1}{2}\star$)

Frequently the construction of spaces out of pieces proceeds by a series of steps where one takes two spaces, say A and B, makes disjoint copies of them, finds closed subspaces C and D that are homeomorphic by some homeomorphism h, and finally glues A and B together using this homeomorphism. For example, one can think of a rectangle as being formed from two right triangles by gluing the latter along the hypotenuse. Of course, there are also many more complicated examples of this sort.

Formally speaking, we can try to model this process by forming the disjoint union $A \coprod B$ and then factoring out by the equivalence relation

$$x \sim y \Longleftrightarrow x = y \quad \textbf{OR}$$

$$x = i_A(a),\ y = i_B(h(a)) \quad \text{for some } a \in A \quad \textbf{OR}$$

$$y = i_A(a),\ x = i_B(h(a)) \quad \text{for some } a \in A.$$

It is an elementary but tedious exercise in bookkeeping to to verify that this defines an equivalence relation (the details are left to the reader!). The resulting quotient space will be denoted by

$$A \bigcup_{h:C\equiv D} B.$$

As a test of how well this approach works, consider the following question:

Scissors and Paste Problem. *Suppose we are given a topological space X and closed subspaces A and B such that $X = A \cup B$. If we take $C = D = A \cap B$ and let h be the identity homeomorphism, does this construction yield the original space X?*

One would expect that the answer is yes, and here is the proof:

Retrieving the original space. Let Y be the quotient space of $A \coprod B$ with respect to the equivalence relation, and let $p : A \coprod B \to Y$ be the quotient map. By the preceding observations, there is a unique continuous map $f : A \coprod B \to X$ such that $f \circ i_A$ and $f \circ i_B$ are the inclusions $A \subset X$ and $B \subset X$ respectively. By construction, if $u \sim v$ with respect to the equivalence relation described above, then $f(u) = f(v)$, and therefore there is a unique continuous map $h : Y \to X$ such that $f = h \circ p$. We claim that h is a homeomorphism. First of all, h is onto because the identities $h \circ p \circ i_A = \text{inclusion}_A$ and $h \circ p \circ i_B = \text{inclusion}_B$ imply that the image contains $A \cup B$, which is all of X. Next, h is 1–1. Suppose that $h(u) = h(v)$ but $u \neq v$, and write $u = p(u')$, $v = p(v')$. The preceding identities imply that h is 1-1 on both A and B, and therefore one of u', v' must lie in A and the other in B. By construction, it follows that the inclusion maps send u' and v' to the same point in X. But this means that u' and v' correspond to the same point in $A \cap B$ so that $u = p(u') = p(v') = v$. Therefore the map h is 1–1. To prove that h is a homeomorphism, it suffices to show that h takes closed subsets to closed subsets. Let F be a closed subset of Y. Then the inverse image $p^{-1}(F)$ is closed in $A \coprod B$. However, if we write write $h(F) \cap A = P$ and $h(f) \cap B = Q$, then it follows that $p^{-1}(F) = i_A(P) \cup i_B(Q)$. Thus $i_A(P) = p^{-1}(F) \cap i_A(A)$ and $i_B(Q) = p^{-1}(F) \cap i_B(B)$, and consequently the subsets $i_A(P)$ and $i_B(Q)$ are closed in $A \coprod B$. But this means that P and Q are closed in A and B respectively, so that $P \cup Q$ is closed in X. Therefore it suffices to verify that $h(F) = P \cup Q$. But if $x \in F$, then the surjectivity of p implies that $x = p(y)$ for some $y \in p^{-1}(F) = i_A(P) \cup i_B(Q)$; if $y \in i_A(P)$ then we have

$$h(x) \quad = \quad h(p(y)) \quad = \quad f(y) \quad = \quad f \circ i_A(y) \quad = \quad y$$

for some $y \in P$, while if $y \in i_B(Q)$ the same sorts of considerations show that $h(x) = y$ for some $y \in Q$. Hence $h(F)$ is contained in $P \cup Q$. On the other hand, if $y \in P$ or $y \in Q$ then the preceding equations for P and their analogs for Q show that $y = h(p(y))$ and $p(y) \in F$ for $y \in P \cup Q$, so that $P \cup Q$ is contained in $h(F)$ as required.■

One can formulate an analog of the scissors and paste problem if A and B are open rather than closed subset of X, and once again the answer is that one does retrieve the original space. The argument is similar to the closed case and is left to the reader as an exercise.

Examples. Many examples for the scissors and paste theorem can be created involving subsets of Euclidean 3-space. For example, as noted before one can view the surface of a cube as being constructed by a sequence of such operations in which one adds a solid square homeomorphic to $[0,1]^2$ to the space constructed at the previous step. Our focus here will involve examples of objects in 4-dimensional space that can be constructed by a single scissors and paste construction involving objects in 3-dimensional space.

1. The *hypersphere* $S^3 \subset \mathbb{R}^4$ is the set of all points (x, y, z, w) whose coordinates satisfy the equation

$$x^2 + y^2 + z^2 + w^2 \quad = \quad 1$$

and it can be constructed from two 3-dimensional disks by gluing them together along the boundary spheres. An explicit homeomorphism

$$D^3 \bigcup_{\mathrm{id}(S^2)} D^3 \longrightarrow S^3$$

can be constructed using the maps

$$f_{\pm}(x, y, z) \quad = \quad \left(x, y, z, \sqrt{1 - x^2 - y^2 - z^2}\right)$$

on the two copies of D^3. The resulting map is well defined because the restrictions of $f_{\pm}$ to S^2 are equal.

2. We shall also show that the Klein bottle can be constructed by gluing together two Möbius strips along the simple closed curves on their edges. Let $g_{\pm} : [-1,1] \to S^1$ be the continuous 1–1 map sending t to $\left(\pm\sqrt{1-t^2}, t\right)$. It then follows that the images $F_{\pm}$ of the maps $\mathrm{id}_{[0,1]} \times [-1,1]$ satisfy $F + \cup F_- = [0,1] \times S^1$ and $F_+ \cap [0,1] \times \{-1,1\}$. If $\varphi : [0,1] \times S^1 \to \mathbf{K}$ is the quotient projection to the Klein bottle, then it is relatively elementary to verify that each of the sets $\varphi(F_{\pm})$ is homeomorphic to the Möbius strip (look at the equivalence relation given by identifying two points if they have the same images under $\varphi \circ g_{\pm}$) and the intersection turns out to be the set $\varphi(F_+) \cap \varphi(F_-)$, which is homeomorphic to the edge curve for either of these Möbius strips.

III.4.A.3 : *Disjoint unions of families of sets*

As in the case of products, one can form disjoint unions of arbitrary finite collections of sets or spaces recursively using the construction for a pair of sets. However, there are also cases where one wants to form disjoint unions of infinite collections, so we shall sketch how this can be done, leaving the proofs to the reader as exercises.

Definition. If A is a set and $\{ X_\alpha \mid \alpha \in A \}$ is a family of sets indexed by A, the *disjoint union* (or set-theoretic sum)

$$\coprod_{\alpha \in A} X_\alpha$$

is the subset of all

$$(x, \alpha) \ \in \ \left(\bigcup_{\alpha \in S} X_\alpha \right) \times A$$

such that $x \in X_\alpha$.

This is a direct generalization of the preceding construction, which may be viewed as the special case where $A = \{1, 2\}$. For each $\beta \in A$ one has an injection map

$$i_\beta : X_\beta \to \coprod_{\alpha \in A} X_\alpha$$

sending x to (x, β); as before, the images of i_β and i_γ are disjoint if $\beta \neq \gamma$ and the union of the images of the maps i_α is all of $\coprod_\alpha X_\alpha$.

Notation. In the setting above, suppose that each X_α is a topological space with topology $\mathbf{T}_\alpha$. Let $\sum_\alpha \mathbf{T}_\alpha$ be the set of all disjoint unions $\coprod_\alpha U_\alpha$ where U_α is open in X_α for each α.

As in the previous discussion, this defines a topology on $\coprod_\alpha X_\alpha$, and the basic properties can be listed as follows:

[**1**] *The family of subsets $\sum_\alpha \mathbf{T}_\alpha$ defines a topology for $\coprod_\alpha X_\alpha$ such that the injection maps i_α are homeomorphisms onto their respective images. the latter are open and closed subspaces of $\coprod_\alpha X_\alpha$, and each injection is continuous, open and closed.*

[**2**] *The closed subsets of $\coprod X_\alpha$ with the disjoint union topology are the sets of the form $\coprod F_\alpha$ where F_α is closed in X_α for each α.*

[**3**] *If each X_α is discrete then so is $\coprod_\alpha X_\alpha$.*

[**4**] *If each X_α is Hausdorff then so is $\coprod_\alpha X_\alpha$.*

[**5**] *If each X_α is homeomorphic to a metric space, then so is $\coprod_\alpha X_\alpha$.*

[**6**] *If for each α we are given a continuous function $f : X_\alpha \to W$ into some fixed space W, then there is a unique continuous map $h : \coprod_\alpha X_\alpha \to W$ such that $h \circ i_\alpha = f_\alpha$ for all α.*

The verifications of these properties are direct extensions of the earlier arguments, and the details are left to the reader.∎

In linear algebra one frequently encounters vector spaces that are isomorphic to direct sums of other spaces but not explicitly presented in this way, and it is important to have simple criteria for recognizing situations of this type. Similarly, in working with topological spaces one frequently encounters spaces that are homeomorphic to disjoint unions but not presented in this way, and in this context it is also convenient to have a simple criterion for recognizing such objects.

INTERNAL SUM RECOGNITION PRINCIPLE. *Suppose that a space Y is a union of pairwise disjoint subspaces X_α, each of which is open and closed in Y. Then Y is homeomorphic to $\coprod_\alpha X_\alpha$.*

Proof. For each $\alpha \in A$ let $j_\alpha : X_\alpha \to Y$ be the inclusion map. By [6] above there is a unique continuous function

$$J : \coprod_\alpha X_\alpha \longrightarrow Y$$

such that $J \circ i_\alpha = j_\alpha$ for all α. We claim that J is a homeomorphism; in other words, we need to show that J is 1–1 onto and open. Suppose that we have $(x_\alpha, \alpha) \in i_\alpha(X_\alpha)$ and $(z_\beta, \beta) \in i_\beta(X_\beta)$ such that $J(x_\alpha, \alpha) = J(z_\beta, \beta)$. By the definition of J this implies $i_\alpha(x_\alpha) = i_\beta(z_\beta)$. Since the images of i_α and i_β are pairwise disjoint, this means that $\alpha = \beta$. Since i_α is an inclusion map, it is 1–1, and therefore we have $x_\alpha = z_\beta$. The proof that J is onto drops out of the identities

$$J\left(\coprod_\alpha X_\alpha\right) \;=\; J\left(\bigcup_\alpha i_\alpha(X_\alpha)\right) \;=\; \bigcup_\alpha J\left(i_\alpha(X_\alpha)\right) \;=\; \bigcup_\alpha j_\alpha(X_\alpha) \;=\; Y\ .$$

Finally, to prove that J is open let W be open in the disjoint union, so that we have

$$W \;=\; \coprod_\alpha U_\alpha$$

where each U_α is open in the corresponding X_α. It then follows that $J(W) = \cup_\alpha U_\alpha$. But for each α we know that U_α is open in X_α and the latter is open in Y, so it follows that each U_α is open in Y and hence that $J(W)$ is open.■

III.4.A.4 : *Constructing topological spaces out of pieces*

In this subsection we shall describe a method for constructing spaces out of relatively complicated data. This procedure is used repeatedly in differential topology and geometry; for example, it provides the framework for constructing spaces of tangent vectors to smooth manifolds in Section III.5 as well as numerous important generalizations.

Disassembly of a space via an open covering. Let X be a topological space, and let $\mathcal{U}$ be an open covering of X consisting of the sets U_α where α lies in some indexing set $\mathbf{A}$. The inclusion map of U_α into X will be denoted by j_α. By results from the preceding subsection, there is a unique continuous function

$$j : \coprod_\alpha U_\alpha \longrightarrow X$$

such that $j \circ i_\alpha = j_\alpha$ for all $\alpha \in \mathbf{A}$. CLAIM: j is an open mapping. — An arbitrary open subset of $\coprod_\alpha U_\alpha$ has the form $\coprod_\alpha V_\alpha$ where V_α is open in U_α (hence also in X). It follows that $j\left(\coprod_\alpha V_\alpha\right)$ is equal to $\cup_\alpha V_\alpha$, and this set is open in X because each V_α is open in X.■

Let $R(\mathcal{U})$ be the equivalence relation on $\coprod_\alpha U_\alpha$ that identifies a and b if and only if $j(a) = j(b)$, and let $\mathbf{p} : \coprod_\alpha U_\alpha \to X^*$ be the projection onto the set of equivalence classes for $R(\mathcal{U})$. Then there is a unique continuous map $J : X^* \to X$ such that $j = J \circ \mathbf{p}$, and by construction J is 1–1 and onto. Since j is open, results on quotient maps in [MUNKRES1] show that *the map J is a homeomorphism.*

III.4.A.5 : *Transition data associated to an open covering*

We are interested in the following problem: *Given an indexed family of topological spaces $\mathcal{U} = \{U_\alpha\}_{\alpha \in \mathbf{A}}$, what additional data are needed to construct an arbitrary topological space X with an open covering topologically equivalent to $\mathcal{U}$?*

The most effective way to analyze this problem is to start with a space X, an open covering $\mathcal{U}$, and the associated continuous open surjection j defined as above. One crucial aspect of understanding the construction of the space X is to study the intersections of two arbitrary open sets in the open covering. Given U_α and U_β in $\mathcal{U}$, define

$$V_{\beta\alpha} \quad = \quad j_\alpha^{-1}(U_\beta) \quad \subset \quad \coprod_\sigma U_\sigma \ .$$

By construction j_α maps $V_{\beta\alpha}$ homeomorphically onto $U_\alpha \cap U_\beta$; likewise, j_β maps $V_{\alpha\beta}$ homeomorphically onto $U_\alpha \cap U_\beta$. Of course this means that $V_{\beta\alpha}$ and $V_{\alpha\beta}$ are homeomorphic, and an explicit homeomorphism

$$\psi_{\beta\alpha} : V_{\beta\alpha} \to V_{\alpha\beta}$$

is given by the following composite:

$$V_{\beta\alpha} \quad = \quad (U_\alpha \cap U_\beta) \times \{\alpha\} \quad \cong \quad (U_\alpha \cap U_\beta) \times \{\beta\} \quad = \quad V_{\alpha\beta}$$

The homeomorphisms $\psi_{\beta\alpha}$ satisfy two basic relations of the form

$$\psi_{\alpha\alpha} \quad = \quad \mathrm{id}(U_\alpha)$$

$$\psi_{\alpha\beta} \quad = \quad \psi_{\beta\alpha}^{-1}$$

as well as a third relation that can be expressed informally as "$\psi_{\gamma\beta} \circ \psi_{\beta\alpha} = \psi_{\gamma\alpha}$." A little care is needed to formulate this precisely because the codomain of $\psi_{\beta\alpha}$ is usually not a subset of the domain of $\psi_{\gamma\beta}$, so it is necessary to be specific about when the composite is definable. This begins with the following observation:

For all α, β, γ in $\mathbf{A}$ the homeomorphism $\psi_{\beta\alpha}$ sends the open subset $V_{\beta\alpha} \cap V_{\gamma\alpha} \subset U_\alpha \times \{\alpha\}$ homeomorphically onto $V_{\alpha\beta} \cap V_{\gamma\beta} \subset U_\beta \times \{\beta\}$.

This is true because the images of the two intersections in X are merely $U_\alpha \cap U_\beta \cap U_\gamma$. Details of this verification are left to the reader.

A precise version of the third relation is then given by

$$\psi_{\gamma\beta}\left(\psi_{\beta\alpha}(x)\right) \quad = \quad \psi_{\gamma\alpha}(x) \quad \text{if} \quad x \in V_{\beta\alpha} \cap V_{\gamma\alpha} \ .$$

The equivalence relation $R(\mathcal{U})$ has an alternate description in terms of the transition homeomorphisms $\psi_{\beta\alpha}$.

PROPOSITION. *Let a and b be points of $\coprod_\sigma U_\sigma$, and let α and β be the indices in $\mathbf{A}$ such that $a \in \mathrm{Image}\ i_\alpha \approx U_\alpha \times \{\alpha\}$ and $b \in \mathrm{Image}\ i_\beta \cong U_\beta \times \{\beta\}$. Then $(a, b) \in R(\mathcal{U})$ if and only if $b = \psi_{\beta\alpha}(a)$.*

Proof. ($\Longrightarrow$) By definition $(a, b) \in R(\mathcal{U})$ means that $j(a) = j(b)$. If this happens, then the common image point lies in $U_\alpha \cap U_\beta$. But this means that $a \in V_{\beta\alpha}$, $b \in V_{\alpha\beta}$ and $b = \psi_{\beta\alpha}(a)$. ($\Longleftarrow$) If $b = \psi_{\beta\alpha}(a)$ then the definition of $\psi_{\beta\alpha}$ implies that $j(a) = j(b)$.

The reason for dwelling on all these definitions and formulas is that they provide the framework for building a space out of pieces. The first step in establishing this is to formulate everything abstractly.

Definition. A *set of topological amalgamation data* is a pair

$$\big(\{Y_\alpha\}, \{\varphi_{\beta\alpha}\}\big)$$

where $\{Y_\alpha\}$ is an indexed family of topological spaces with indexing set $\mathbf{A}$ and $\{\varphi_{\beta\alpha}\}$ is an indexed family of homeomorphisms with indexing set $\mathbf{A} \times \mathbf{A}$ such that the following conditions hold:

(i) For every α and β the map $\varphi_{\beta\alpha}$ is a homeomorphism from an open subset $W_{\beta\alpha}$ of Y_α to an open subset $W_{\alpha\beta}$ of Y_β.

(ii) For every α the map $\varphi_{\alpha\alpha}$ is the identity map for Y_α, and for every α and β the map $\varphi_{\alpha\beta}$ is the inverse homeomorphism to $\varphi_{\beta\alpha}$.

(iii) For every α, β and γ the map $\varphi_{\beta\alpha}$ sends $W_{\beta\alpha} \cap W_{\gamma\alpha} \subset Y_\alpha$ homeomorphically onto $W_{\alpha\beta} \cap W_{\gamma\beta} \subset Y_\beta$, and $\varphi_{\gamma\beta}\,(\,\varphi_{\beta\alpha}(y)\,) = \varphi_{\gamma\alpha}(y)$ for all $y \in W_{\beta\alpha} \cap W_{\gamma\alpha}$.

The functional identities described above are called **cocycle formulas** or something similar in Conlon's book and numerous other places (the key word is "cocycle").

We have stated the definition so that the preceding construction defines a set of topological amalgamation data associated to an open covering of a topological space.

There is a corresponding concept of isomorphism; for convenience we shall assume that we have two sets of topological amalgamation data with the same indexing set (QUESTION: *What modifications are necessary if we do not make this assumption?*). Given two such structures

$$\big(\{Y_\alpha\}, \{\varphi_{\beta\alpha}\}\big) \qquad\qquad \big(\{U_\alpha\}, \{\psi_{\beta\alpha}\}\big)$$

an isomorphism between them consists of an indexed family of homeomorphisms $\{h_\alpha : Y_\alpha \to U_\alpha\}$ such that

(a) for every α and β the maps h_α and h_β send the domain and codomain of $\varphi_{\beta\alpha}$ homeomorphically onto the domain and codomain of $\psi_{\beta\alpha}$ respectively,

(b) for every α and β and for every y in the domain of $\varphi_{\beta\alpha}$ we have the following commutativity relation:

$$\varphi_{\beta\alpha}\left(h_\alpha(y)\right) \quad = \quad h_\beta\left(\varphi_{\beta\alpha}(y)\right)$$

We are now ready to state the result we want on building a space out of pieces:

TOPOLOGICAL REALIZATION THEOREM. *If* $\mathbf{Y} = (\,\{Y_\alpha\}, \{\varphi_{\beta\alpha}\}\,)$ *is a set of topological amalgamation data then there is a space* X *and an open covering* $\mathcal{U}$ *of* X *such that* $\mathbf{Y}$ *is isomorphic to the topological amalgamation data associated to* $\mathcal{U}$. *The space* X *is uniquely determined up to homeomorphism.*

Proof. ($\star$) Let Y be the disjoint union $\coprod_\sigma Y_\sigma$, and define a binary relation $R(\mathbf{Y})$ on Y by stipulating that (a, b) lies in the graph of $R(\mathbf{Y})$ if and only if $b = \varphi_{\beta\alpha}(a)$, where $a \in Y_\alpha$ and $b \in Y_\beta$.

The first order of business is to verify that $R(\mathbf{Y})$ is an equivalence relation. The relation is *reflexive* because the first part of property (ii) in the definition implies that $a = \varphi_{\alpha\alpha}(a)$. Similarly, the relation is *reflexive* because the second part of property (ii) in the definition shows that $a =$

$\varphi_{\alpha\beta}(b)$ if $b = \varphi_{\beta\alpha}(a)$. Finally, to verify that the relation is also *transitive*, let a, b and c satisfy $b = \varphi_{\beta\alpha}(a)$ and $c = \varphi_{\gamma\beta}(b)$. It follows that b lies in the intersection $W_{\alpha\beta} \cap W_{\gamma\beta}$, and therfore by the first part of property (iii) it follows that a lies in $W_{\beta\alpha} \cap W_{\gamma\alpha}$. Therefore $\varphi_{\gamma\alpha}(a)$ is defined, and by the second part of property (iii) we have

$$\varphi_{\gamma\alpha}(a) \quad = \quad \varphi_{\gamma\beta}(\varphi_{\beta\alpha}(a))$$

and using the assumptions on a, b and c we may rewrite the right hand side as

$$\varphi_{\gamma\beta}(b) \quad = \quad c$$

so that $c = \varphi_{\gamma\alpha}(a)$, which means that (a, c) lies in the graph of $R(\mathbf{Y})$ and consequently the latter is an equivalence relation as expected.

Let X be the set of equivalence classes of $R(\mathbf{Y})$ with the quotient topology, and for each α let k_α be the composite of the quotient map $p : Y \to X$ with the inclusion $i_\alpha : Y_\alpha \to Y$. We claim that for each α the map h_α is 1–1, continuous and open. Continuity follows immediately because h_α is a composite of two continuous functions. If a and a' lie in Y_α, then their images in X are equal if and only if $a' = \varphi_{\alpha\alpha}(a)$. But $\varphi_{\alpha\alpha}$ is the identity map, so we must have $a = a'$. Finally, to show that k_α is open, let N be an open subset of Y_α; to show that $k_\alpha(N)$ is open in X we need to show that $p^{-1}[k_\alpha(N)]$ is open in Y. But

$$p^{-1}[k_\alpha(N)] \quad \cong \quad \coprod_\beta \varphi_{\beta\alpha}^{-1}[N]$$

and the latter is open in Y by the continuity of the maps $\varphi_{\beta\alpha}$. Therefore k_α is open as claimed.

If we set $U_\alpha = k_\alpha(Y_\alpha)$ then $\mathcal{U} = \{U_\alpha\}$ is an open covering of X. It is left as an exercise for the reader to verify that the original set of topological amalgamation data is isomorphic to the corresponding data set associated to $\mathcal{U}$.

It remains to prove that X is unique up to homeomorphism. Suppose there are spaces X and X' with open coverings $\mathcal{U}$ and $\mathcal{U}'$ such that $\mathbf{Y}$ is isomorphic to the sets of topological amalgamation data associated to both $\mathcal{U}$ and $\mathcal{U}'$. By transitivity of isomorphisms it follows that the data sets associated to the two open coverings are isomorphic, so it suffices to show that X and X' are homeomorphic if the data sets are isomorphic.

For each α let $h_\alpha : U_\alpha \to U'_\alpha$ be the homeomorphism associated to the isomorphism of amalgamation data. We then have an corresponding homeomorphism:

$$\coprod_\alpha h_\alpha : \coprod_\alpha U_\alpha \longrightarrow \coprod_\alpha U'_\alpha$$

Let $\mathbf{p}$ and $\mathbf{p}'$ be the canonical quotient maps from $\coprod_\alpha U_\alpha$ and $\coprod_\alpha U'_\alpha$ to X and X' (respectively) as defined at the beginning of this writeup. By the commutativity relation from (b) in the definition of an isomorphism it follows that $\coprod_\alpha h_\alpha$ passes to a continuous map $h : X \to X'$ of these quotient spaces. We claim that h is a homeomorphism.

The continuity of h is already known, and the next step is to prove that h is onto. If $z \in X'$, choose α so that $z \in U'_\alpha$ (at least one exists because we have an open covering of X'). If $t \in Y'_\alpha$ maps to z under the quotient map $\mathbf{p}'$, then by construction we have that

$$z \quad = \quad h\left(j_\alpha\left(h_\alpha^{-1}(t)\right)\right)$$

showing that the arbitrary point z lies in the image of h.

To show that h is 1–1, it suffices to show that if $h(\mathbf{p}(a)) = h(\mathbf{p}(b))$ then $\mathbf{p}(a) = \mathbf{p}(b)$. Since h is the map of quotient spaces determined by $\coprod_\sigma h_\sigma$ we have the commutativity relation

$$h^\circ \mathbf{p} \quad = \quad \mathbf{p}'^\circ \left(\coprod_\sigma h_\sigma \right)$$

and thus if $a \in U_\alpha$ and $b \in U_\beta$ the hypothesis $h(\mathbf{p}(a)) = h(\mathbf{p}(b))$ can be rewritten as $\mathbf{p}'(h_\alpha(a)) = \mathbf{p}'(h_\beta(b))$. But this means that $\psi'_{\beta\alpha}(h_\alpha(a)) = h_\beta(b)$. On the other hand, by the commutativity relation in part (b) of the definition of an isomorphism we know that $\psi'_{\beta\alpha}{}^\circ h_\alpha = h_\beta{}^\circ \psi_{\beta\alpha}$, and this in turn implies that

$$h_\beta(b) \quad = \quad h_\beta\left(\psi_{\beta\alpha}(a)\right).$$

Since h_β is a homeomorphism this implies that $b = \psi_{\beta\alpha}(a)$, which means that $\mathbf{p}(b) = \mathbf{p}(a)$ and proves that h is 1–1.

The last step is to prove that h is open. This will be a special case of the following more general result:

LEMMA. *Let X and Y be topological spaces, let R and S be equivalence relations on X and Y respectively, let $p : X \to X/R$ and $q : Y \to Y/S$ be the corresponding quotient space projections, and suppose that f is a continuous map from X to Y that is $1-1$ onto and takes R-equivalent points in X to S-equivalent maps in Y. Denote the associated map of quotient spaces from X/R to Y/S by h. If f, p and q are open mappings then so is h.*

Proof. Suppose that U is open in X/R. Then $h(U)$ is open in Y/S if and only if $q^{-1}[h(U)]$ is open in Y. Since f is continuous, open and onto, it follows that $q^{-1}[h(U)]$ is open in Y if and only if

$$f^{-1}\left[q^{-1}[h(U)]\right] \quad = \quad p^{-1}\left[h^{-1}[h(U)]\right]$$

is open in X. Since h is 1–1 and onto it follows that $U = [h^{-1}[h(U)]$, and therefore the right hand side of the displayed equation is merely the set $p^{-1}[U]$, which is open by the continuity of p. It follows that the map h is open as asserted.∎

As noted above, this completes the proof of the Realization Theorem.∎

One recurrent question is whether the space constructed from amalgamation data is Hausdorff if all the pieces are. Perhaps the simplest examples yielding non-Hausdorff spaces are given by taking $Y_1 = Y_2 = \mathbb{R}^n$, $U_{21} = U_{12} = \mathbb{R}^n - \{0\}$, and $\varphi_{12} = \varphi_{21}$ to be the identity map. The space constructed from these data is the non-Hausdorff Forked Line that we first introduced in Section I.1.

In contrast, the next result essentially says that problems with pairs of subspaces are the only things can prevent the constructed space from being Hausdorff.

PROPOSITION. *Let $\mathbf{Y} = (\{Y_\alpha\}, \{\varphi_{\beta\alpha}\})$ be a set of topological amalgamation data, and let X be the space with an open covering $\mathcal{U}$ of X such that $\mathbf{Y}$ is isomorphic to the topological amalgamation data associated to $\mathcal{U}$. Then the space X is Hausdorff if and only if each Y_α is Hausdorff and for each β and γ the space*

$$X_{\beta\alpha} \quad = \quad Y_\alpha \cup_{\varphi_{\beta\alpha}: W_{\alpha\beta} \equiv W_{\beta\alpha}} Y_\beta$$

is Hausdorff.

Proof. The conditions are clearly necessary. To prove they are sufficient, let u and v be distinct points of X. If they both lie in some Y_α, then they have disjoint neighborhoods in Y_α because the latter is Hausdorff. If one lies in, say, Y_β and the other in Y_γ, then the points have disjoint neighborhoods in $X_{\beta\alpha}$ because this space is Hausdorff. In either case, two distinct points have disjoint neighborhoods as required.∎

III.4.B : Smooth amalgamation

One can adapt much of the discussion in III.A to obtain a comparable theory for smooth manifolds.

III.4.B.1 : *Smooth structures on disjoint unions*

The first step is to give a smooth version of the disjoint union construction. This turns out to be extremely straightforward.

SMOOTH DISJOINT UNIONS. *Let $\{ (M_\alpha, \mathcal{A}_\alpha) \}$ be a family of smooth manifolds, and let $\coprod_\alpha M_\alpha$ be their disjoint union. Then $\coprod_\alpha \mathcal{A}_\alpha$ defines a smooth atlas for $\Sigma = \coprod_\alpha M_\alpha$ such that*

(i) the injections $i_\alpha : M_\alpha \to \Sigma$ are smooth mappings,

(ii) if $(L, \mathcal{E}$ is a smooth manifold and $f : \Sigma \to L$ is continuous, then f is smooth if and only if each composite $f \circ i_\alpha$ is smooth.

In Section I.2 we made a standing hypothesis of second countability for the manifolds considered in this course. Since a disjoint union of of nonempty spaces is second countable only if the number of such spaces is $\leq \aleph_0$ (see page 3 of the file

`http://math.ucr.edu/~res/math205A/solutions5.pdf`

for details) the standing hypothesis implies that the family $\{ (M_\alpha, \mathcal{A}_\alpha) \}$ should be assumed to be countable.

Proof. The images of the charts in $\coprod_\alpha \mathcal{A}_\alpha$ form an open covering for $\coprod_\alpha M_\alpha$; to see that $\coprod_\alpha \mathcal{A}_\alpha$ determines a smooth atlas, note that the images of two charts (U, h) and (V, k) intersect nontrivially only if both belong to one of the subfamilies $\mathcal{A}_\ell$, and because of this all nontrivial transition maps (*i.e.*, those defined on nonempty open sets) will be smooth. Therefore $\coprod_\alpha \mathcal{A}_\alpha$ is a smooth atlas (however, as noted in the exercises it is usually not a maximal atlas).

Smoothness of the injection maps i_α may be established by noting that for each smooth chart (U, h) in $\coprod_\alpha \mathcal{A}_\alpha$ the local map "$(i_\alpha \circ h)^{-1} \circ h$" is equal to the id_U. To verify property (ii), first observe that the $(\Longrightarrow)$ implication follows because composites of smooth maps are smooth. Conversely, if each of the composites $h \circ i_\alpha$ is smooth, then for all charts (U, h) in $\coprod_\alpha \mathcal{A}_\alpha$ and (V, k) in $\mathcal{E}$ satisfying $f \circ i_\alpha \circ h(U) \subset k(V)$ we know that "$k^{-1} \circ h \circ h$" is smooth; but this implies that h itself is also smooth.∎

In Section III.A we gave a result called the Internal Sum Recognition Principle for recognizing spaces that are equivalent to disjoint unions; as noted there, similar situations arise naturally in linear algebra where it is often important to find internal direct sum structures on vector spaces. We would like to state and prove a corresponding recognition principle for smooth manifolds. Before doing so we shall a prove a preliminary result of independent interest.

LEMMA. *In the notation of the previous result, for each smooth manifold $(M_\beta, \mathcal{A}_\beta)$ in the collection $\{ (M_\alpha, \mathcal{A}_\alpha) \}$ the map i_β defines a diffeomorphism i'_β from M_β to $i_\beta(M_\beta)$.*

Proof. We already know that the associated map i'_β is a homeomorphism from M_β to $i_\beta(M_\beta)$, and the considerations of Section III.2 combine with conclusion (i) in the previous result to imply that i'_β is smooth. We shall prove that the inverse is smooth by presenting the inverse as a composite of smooth maps.

By the results for topological disjoint unions, one can define a continuous map $q : \coprod_\alpha \to M_\beta$ such that $q \circ i_\beta$ = identity and $q \circ i_\alpha$ = constant if $\alpha \neq \beta$. By construction each of the maps $q \circ i_\alpha$ is smooth, and therefore conclusion (ii) of the previous result implies that q is smooth. Direct computation then shows that the smooth map $q | i_\beta(M_\beta)$ is an inverse (hence **the** inverse) to $i_\beta(M_\beta)$.■

The lemma leads directly to the following simple criterion for recognizing smooth manifolds that are diffeomorphic to disjoint unions:

SMOOTH INTERNAL SUM RECOGNITION PRINCIPLE. *Suppose that a smooth manifold $(N, \mathcal{B})$ is a union of pairwise disjoint open subspaces M_α. Then N is homeomorphic to $\coprod_\alpha M_\alpha$.*

Proof. For each $\alpha \in A$ let $j_\alpha : M_\alpha \to N$ be the inclusion map. By the previous results for topological and smooth disjoint unions, there is a unique smooth function

$$J : \coprod_\alpha M_\alpha \longrightarrow N$$

such that $J \circ i_\alpha = j_\alpha$ for all α. We claim that J is a diffeomorphism; by the topological version of the internal sum recognition principle we know that J is a homeomorphism.

To show that J is a diffeomorphism, it only remains to show that J^{-1} is smooth. It will suffice to show that the restriction of J^{-1} to each open subset M_α is smooth. A direct examination of the definitions shows that $J^{-1} | M_\alpha$ is equal to i_α, which we know is smooth, and therefore it follows that J^{-1} must also be smooth.■

III.4.B.2 : *Constructing smooth manifolds out of pieces*

For smooth manifolds one can also give a condition for realizing the amalgamation data by a smooth atlas.

SMOOTH REALIZATION THEOREM. *In the setting of the Topological Realization Theorem above, suppose that the spaces Y_α are all open subsets of $\mathbb{R}^n$, the maps $\varphi_{\beta\alpha}$ are all diffeomorphisms, and the associated space X is Hausdorff and second countable. Then X is a second countable topological n-manifold, and it has a smooth atlas $\{(U_\alpha, h_\alpha)\}$ such that the transition maps "$h_\beta^{-1} h_\alpha$" are equal to $\varphi_{\beta\alpha}$ for all α and β.*

Notation. In the situation of this result, if $\mathcal{A}$ is the original set of amalgamation data then the corresponding smooth atlas on the constructed space X will be called the *associated smooth atlas* for the amalgamation data.

Proof of the theorem. We shall use the notation in the proof of the Topological Realization Theorem (*q.v.*). The space X is a topological manifold because it has an open covering consisting of topological manifolds (in fact, open subsets in $\mathbb{R}^n$). Consider the atlas consisting of the pairs (Y_α, k_α) described in the proof of the Topological Realization Theorem. The transition maps "$k_\beta^{-1} \circ k_\alpha$" for this atlas are equal to the diffeomorphisms $\varphi_{\beta\alpha}$ by construction.■

III.5 : Tangent spaces and vector bundles

(Conlon, §§ 3.3–3.4)

A basic idea underlying the theory of smooth manifolds is that such objects can be studied using a mixture of techniques from multivariable calculus and point set topology. We have already discussed some constructions for topological spaces for which there are similar constructions on smooth manifolds in at least some cases, including finite products, covering space projections, submanifolds, quotient constructions related to covering space projections and disjoint sums.

Despite these similarities, there are also clear differences between what one can do for topological spaces as opposed to smooth manifolds. In particular, there are numerous constructions on topological spaces that do not work at all for smooth manifolds, but on the other hand there are also some important constructions for smooth manifolds that cannot be carried out for topological spaces. The **tangent bundle** of a smooth manifold is a fundamental example of this sort.

III.5.1 : *Definitions and examples*

The definition of the tangent bundle requires some digressions, so it seems best to begin with a description of what we want. For an open subsubset U of $\mathbb{R}^n$ we defined the space of all tangent vectors to points of U to be the product $U \times \mathbb{R}^n$, the idea being that for each $x \in U$ the space $\{x\} \times \mathbb{R}^n$ can be viewed as a space of tangent vectors at x (or as a a physicist might say, vectors whose point of application is x). Similarly, if we are given a smooth n-manifold $(M, \mathcal{A})$ and a point p in M, we would like to describe a smooth manifold $T(M)$ such that for each $p \in M$ it contains an n-dimensional vector space $T_p(M)$ of *tangent vectors* to p in M, and such that $T(M)$ is the union of these vector spaces for all $p \in M$; for the record, we would also like these vector spaces to be pairwise disjoint. The space $T_p(M)$ should be defined so that its elements can be viewed as tangent vectors for smooth curves $\varphi : (-\varepsilon, \varepsilon)$ satisfying $\varphi(0) = p$; in other words, for each vector $\mathbf{v} \in T_p(M)$ one can find a φ of this sort so that it makes sense to say $\varphi'(0) = \mathbf{v}$.

If M is open in $\mathbb{R}^n$ our previous construction fulfills these requirements. As usual, the best test case for extending the definition is the standard 2-sphere in Euclidean 3-space.

There are two possible approaches, and they lead to the same answer. On one hand, in classical solid geometry one speaks about the tangent plane to a point on a sphere as the plane perpendicular to the radius at the point of contact. This is good for looking at a single tangent plane, but classical tangent planes generally intersect in a line and *we want our tangent planes at different points to* *be pairwise distinct*. One way of creating an object that fulfills this requirement and still leads to the classical notion of tangent plane is to view the tangent space for S^2 to be the set of all points $(x, v) \in S^2 \times \mathbb{R}^3$ such that $|x| = 1$ (*i.e.*, it lies on S^2) and y is perpendicular to x. The classical tangent plane to x will then be the set of all points of the form $x + y$ where y is perpendicular to x.

A second way of approaching this is through the following elementary result:

PROPOSITION. *Let $x \in S^2$ and $y \in \mathbb{R}^3$. Then there is a smooth curve $\varphi : (-\varepsilon, \varepsilon) \to S^2$ such that $\varphi(0) = x$ and $\varphi'(0) = y$ if and only if $\langle x, y\rangle = 0$, where $\langle\ ,\ \rangle$ denotes the usual inner product on $\mathbb{R}^3$.*

In fact, this all generalizes to level sets of regular values. If $f : \mathbb{R}^n \to \mathbb{R}^m$ is a smooth map (where $n > m$ as usual) and y is a nontrivial regular value of f, then the tangent space of level set $L = f^{-1}(\{y\})$ can be taken to be the set of all points $(u, \mathbf{v}) \in L \times \mathbb{R}^n$ such that $Df(u)\mathbf{v} = 0$. Since f is a regular value the dimension of the kernel of $Df(u)$ is $(n - m)$ for all $u \in \mathbb{R}^n$. The preceding proposition extends directly to such level sets with this definition of the tangent space. In particular, for the unit sphere we are looking at the set of all points where $f(x) = 1$, where $f(x) = |x|^2$, and in this case $Df(x)y = 2\langle x, y\rangle$.

By the Theorem on Level Sets in Section III.1, there is an atlas of smooth charts (U_α, h_α) for L such that each $j \circ h_\alpha$ is smooth. Suppose that $\in L$ is chosen so that $x \in h_\alpha(U_\alpha) \cap h_\beta(U_\beta)$, and let $\mathbf{v}$ be a vector in the kernel of $Df(x)$. Then one can use the coordinate charts to construct smooth curves $\Gamma_\alpha : (-\varepsilon, \varepsilon) \to U_\alpha$ and $\Gamma_\beta : (-\varepsilon, \varepsilon) \to U_\beta$ such that $h_\alpha \circ \Gamma_\alpha = h_\beta \circ \Gamma_\beta$, $h_\alpha(\Gamma_\alpha(0)) = h_\beta(\Gamma_\beta(0))$ and if $\Gamma : (-\varepsilon, \varepsilon) \to \mathbb{R}^n$ is the the associated smooth curve in Euclidean m-space then $\Gamma'(0) = \mathbf{v}$.

FUNDAMENTAL QUESTION. *What is the relationship between the tangent vectors $\Gamma'_\alpha(0)$ and $\Gamma'_\beta(0)$?*

ANSWER. By construction we have that Γ_β is equal to "$h_\beta^{-1}h_\alpha$" $\circ \Gamma_\alpha$, and therefore by the Chain Rule the tangent vector $\mathbf{w}$ at $u = \Gamma_\alpha(0)$ is identified with the tangent vector D"$h_\beta^{-1}h_\alpha$"$(u)\mathbf{w}$ at "$h_\beta^{-1}h_\alpha$"$(u) = \Gamma_\beta(0)$.■

All of these considerations are part of the following result:

THEOREM. *Let $n > m$ and let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a smooth map such that y is a nontrivial regular value f (i.e., there is some x so that $f(x) = y$), and let $L = f^{-1}(\{y\})$. Then the tangent space to L, consisting of all $(x, y) \in L \times \mathbb{R}^n$ such that $Df(x)y = 0$, is a smooth manifold, and if $\mathcal{A} = \{(U_\alpha, h_\alpha)\}$ is a smooth atlas of the type described above, then there is a smooth atlas for the tangent space of L having the form $\{(U_\alpha \times \mathbb{R}^{n-m}, H_\alpha)\}$ where $H_\alpha(x, \mathbf{v}) = (h_\alpha(x), Dh_\alpha(x)\mathbf{v})$.*

The transition maps are smooth because they are given by the formula "$H_\beta^{-1}H_\alpha$"$(x, \mathbf{v}) =$ ("$h_\beta^{-1}h_\alpha$" $(x), D[$"$h_\beta^{-1}h_\alpha$"$](x)\mathbf{v})$.■

III.5.2 : *General construction for the tangent bundle*

Motivated by the level sets example, we would like to construct the tangent space of an arbitrary smooth manifold $(M, \mathcal{A})$ out of the following data:

For each chart (U_α, h_α) in the maximal atlas $\mathcal{A}$, define Y_α to be $U_\alpha \times \mathbb{R}^n$. Following standard practice we define $V_{\beta\alpha} \subset U_\alpha$ to be the open subset

$$h_\alpha^{-1}\big(\, h_\beta(U_\beta)\,\big)$$

and let $\psi_{\beta\alpha} : V_{\beta\alpha} \to V_{\alpha\beta}$ be the usual transition map "$h_\beta^{-1} \circ h_\alpha$" that is a diffeomorphism because $\mathcal{A}$ is a smooth atlas. We then take the open subset $W_{\beta\alpha}$ to be the product $V_{\beta\alpha} \times \mathbb{R}^n$ define mappings

$$\varphi_{\beta\alpha} : V_{\beta\alpha} \times \mathbb{R}^n \longrightarrow V_{\alpha\beta} \times \mathbb{R}^n$$

by the following formula:

$$\varphi_{\beta\alpha}(x, \mathbf{v}) \quad = \quad \big(\, \psi_{\beta\alpha}(x),\, D\psi_{\beta\alpha}(x)\mathbf{v}\,\big)$$

PROPOSITION. *The preceding data* $(\,\{Y_\alpha\},\, \{\,\varphi_{\beta\alpha}\,\}\,)$ *define a set of topological amalgamation data.*

Proof. In order to show that we have a set of amalgamation data it is necessary to

- (i) verify that the maps $\varphi_{\beta\alpha}$ are homeomorphisms,
- (ii) check that the cocycle formulas hold.

In fact, the first of these statements is implicitly contained in the second, so it $\varphi_{\beta\alpha} = \varphi_{\alpha\beta}^{-1}$), so it suffices to check that $\varphi_{\gamma,\gamma} =$ identity and "$\varphi_{\gamma\beta} \circ \varphi_{\beta\alpha}$'$= \varphi_{\gamma\alpha}$. These identities may be checked as follows:

- (i) $\varphi_{\alpha\alpha}$ is the identity because $\psi_{\alpha\alpha} =$"$h_\alpha^{-1} h_\alpha$" is the identity and the derivative of an identity map is just the identity matrix.
- (ii) To see that $\varphi_{\beta\alpha}$ and $\varphi_{\alpha\beta}$ are inverse to each other, it suffices to calculate the composites explicitly using the fact that the inverse function identity, $\psi_{\beta\alpha}^{-1} =$["$h_\beta^{-1} h_\alpha$"]$^{-1}$ equals $\psi_{\alpha\beta} =$"$h_\alpha^{-1} h_\beta$", implies $D\psi_{\beta\alpha}(x)^{-1} =$[$D$"$h_\beta^{-1} h_\alpha$"]$(x)^{-1}$ is equal to $D\psi_{\alpha\beta}(x) =$ D"$h_\alpha^{-1} h_\beta$"(x) for all x.
- (iii) To see the composition relation, it suffices to calculate the composites explicitly using the fact that D"$h_\gamma^{-1} h_\alpha$" is the matrix product of equals D"$h_\gamma^{-1} h_\beta$" and D"$h_\beta^{-1} h_\alpha$" by the Chain Rule.

This completes the verification that we have a set of topological amalgamation data.■

By the preceding result and the Topological realization theorems of Section III.4, there is a topological space $T(M)$ realizing the given data. If we choose n such that M is an n-manifold, it follows immediately that every point in $T(M)$ has an open neighborhood that is homeomorphic to an open subset of $\mathbb{R}^{2n}$. The transition maps $\varphi_{\beta\alpha}$ are all smooth (hence diffeomorphisms), and therefore $T(M)$ will be a topological manifold if it is Hausdorff. In addition to this, we really need to verify that $T(M)$ satisfies the standing hypothesis of second countability from Section I.2 (assuming that it holds for M itself!).

Our verification of the Hausdorff and second countability properties for $T(M)$ will depend upon the following fundamental result that is of considerable importance in its own right:

PROPOSITION. *In the setting described above, there is a continuous open surjection* $\tau_M : T(M) \to M$ *such that for each smooth chart* (U_α, h_α) *in the maximal atlas* $\mathcal{A}$ *for* M *the, following conclusions hold:*

(*i*) *The inverse image* $\tau_M^{-1}(\,h_\alpha(U_\alpha)\,)$ *is homeomorphic to* $h_\alpha(U_\alpha)\times\mathbb{R}^n$.

(*ii*) *One can choose the homeomorphism* η *in* (*i*) *so that* $\tau_M\circ\eta(x,\mathbf{v}) = x$ *for all* x *and* $\mathbf{V}$.

Proof. For each α in the indexing set for $\mathcal{A}$, define t_α on $Y_\alpha = U_\alpha\times\mathbb{R}^n$ by $t_\alpha(x,\mathbf{v}) = h_\alpha(x)$. We claim that these maps fit together to define a continuous function on $T(M)$. This will hold if and only if the maps satisfy the following consistency condition with respect to the transition maps:

$$t_\alpha(x,\mathbf{v}) \quad = \quad t_\beta\big(\,\psi_{\beta\alpha}(x), D\psi_{\beta\alpha}(x)\mathbf{v}\,\big)$$

By construction the left hand side is equal to $h_\alpha(x)$ and the right hand side is equal to $h_\beta\,(\,\psi_{\beta\alpha}(x)\,)$. Since the latter is equal to $h_\alpha(x)$, it follows that the locally defined maps fit together to form a continuous map from $T(M)$ to M.

To see that the map τ_M is onto, note first that an arbitrary element of M is expressible as $h_\alpha(x)$ for some α and $x\in U_\alpha$; if $k_\alpha : Y_\alpha \to T(M)$ is the standard 1–1 continuous open map constructed in the realization theorem, then it follows immediately that $h_\alpha(x) = \tau_M\circ k_\alpha(x,\mathbf{0})$, so τ_M is onto. We shall next prove that τ_M is open. Since $\tau(\cup_\beta\, W_\beta) = \cup_\beta\, \tau_M(W_\beta)$ and the sets $k_\alpha(Y_\alpha)$ form an open covering of $T(M)$, it suffices to show that τ_M maps open subsets of $k_\alpha(Y_\alpha)$ to open subsets of $h_\alpha(U_\alpha)$ and since k_α is a homeomorphism onto the open subset $k_\alpha(Y_\alpha)$ it suffices to prove that t_α is open for each α. If $\pi : Y_\alpha = U_\alpha\times\mathbb{R}^n$ denotes projection onto the first coordinate then $t_\alpha = h_\alpha\circ\pi$. Both factors in this composite are open mappings, and therefore t_α is also open; it follows that τ_M is also open.

To prove conclusions (*i*) and (*ii*) it suffices to show that the inverse image of $h_\alpha(U_\alpha)$ is equal to $k_\alpha(Y_\alpha)$. By construction this set is contained in the inverse image. Suppose now that we are given some point $k_\beta(y,\mathbf{w})\in T(M)$ such that $\tau_M\circ k_\beta(y,\mathbf{w})\in h_\alpha(U_\alpha)$. By the definitions of the functions it follows that $h_\beta(y)\in h_\alpha(U_\alpha)$; the latter in turn implies that $\varphi_{\alpha\beta}(y,\mathbf{w})$ is defined and that

$$k_\beta(y,\mathbf{w}) = k_\alpha\circ\varphi_{\alpha\beta}(y,\mathbf{w})$$

so that $k_\beta(y,\mathbf{w})$ lies in the image of k_α. It follows that $\tau_M^{-1}(h_\alpha(U_\alpha))$ is contained in $k_\alpha(Y_\alpha)$ and hence by the second sentence of this paragraph the two sets must be equal.■

By construction the maps $\varphi_{\beta\alpha}$ are all diffeomorphisms, so we are reduced to showing two things; namely, the space $T(M)$ constructed from the preceding amalgamation data is second countable if M is, and it is always Hausdorff.

How does this help with proving that $T(M)$ is Hausdorff and second countable? It will suffice to combine the preceding observation with the following straightforward results in point set topology:

PROPOSITION. *Let* X *and* Y *be topological spaces, and let* $g : X\to Y$ *be a continuous map such that each point* $y\in Y$ *has an open neighborhood* V *for which* $g^{-1}(V)$ *is homeomorphic to a product* $V\times F$, *for some space* F, *by a homeomorphism* $h : V\times F\to g^{-1}(V)$ *satisfying* $g(h(v,z)) = v$ *for all* v *and* z.

(A) *If* Y *and* F *are second countable then so is* X.

(B) *If* Y *and* F *are both Hausdorff then so is* X.

Sketch of Proof. (A) Since Y is second countable, there is a countable open covering $\{V_j\}$ where the open sets satisfy the local hypothesis. Each of the open subsets is also second countable, and a product of second countable sets is second countable, so X is a countable union of the second countable spaces $g^{-1}(V_j)$. But if a space can be expressed as a countable union of second countable open subsets, it must also be second countable (why?).

(B) Suppose that $x_1 \neq x_2$ in X. If $g(x_1) \neq g(x_2)$ then there are disjoint neighborhoods U_1 and U_2 of these image points in Y, and the inverse images $g^{-1}(U_1)$ and $g^{-1}(U_2)$ must be disjoint neighborhoods of x_1 and x_2. On the other hand, if $g(x_1) = g(x_2)$ let V be an open neighborhood of this point as described in the hypothesis of the theorem. The inverse image of this neighborhood is homeomorphic to $V \times F$ for some Hausdorff space F, and under this homeomorphism x_i corresponds to (v_i, z_i) where $v_1 = v_2$ but $z_1 \neq z_2$. Choose disjoint neighborhoods W_1 and W_2 for z_1 and z_2 in F such that $z_i \in W_i$ for $i = 1, 2$. Then the images $h(V \times W_1)$ and $h(V \times W_2)$ are open subsets of $g^{-1}(V)$ that are disjoint neighborhoods of x_1 and x_2 in X.■

COROLLARY. *The space $T(M)$ is a (second countable) smooth manifold and $\tau_M : T(M) \to M$ is a smooth map.*

Proof. We had reduced the proof that $T(M)$ was a second countable topological manifold to showing that it is Hausdorff and second countable. The preceding two propositions imply these facts. Since we had also shown that the transition maps for the amalgamation data are smooth, it follows that the amalgamation data yield a smooth atlas for $T(M)$.

The smoothness assertion follows because τ_M maps each set $k_\alpha(Y_\alpha)$ to $h_\alpha(U_\alpha)$ and the local map "$k_\alpha^{-1} \circ \tau_M \circ h_\alpha$" is just the projection map from $U_\alpha \times \mathbb{R}^n$ to U_α. Since this map is smooth, it follows that τ_M is smooth.■

We shall conclude this subsection with two remarks on atlases for $T(M)$.

The atlas we have constructed for $T(M)$ is **not** a maximal atlas for the tangent space. Consider the case $M = \mathbb{R}^n$. If we take $\mathcal{A}$ to be the atlas whose only chart is the identity map, then we see that $T(M) \cong M \times \mathbb{R}^n$ such that τ_M corresponds to projection onto the first factor (use the proposition). The charts in the standard atlas for $T(M)$ all map onto vertical strips of the form $W \times \mathbb{R}^n$, and of course there are many smooth charts on $T(M) \cong M \times \mathbb{R}^n \cong \mathbb{R}^{2n}$ that do not have this form.

In many situations the following observation on smooth atlases for $T(M)$ is useful:

PROPOSITION. *Let $(M, \mathcal{A})$ be a smooth manifold, and let $\mathcal{B}$ be a subatlas of $\mathcal{A}$. Given a smooth chart (U_α, h_α) in $\mathcal{A}$, let $(U_\alpha \times \mathbb{R}^n, k_\alpha)$ be the associated smooth chart for $T(M)$. Then the set $T(\mathcal{B})$ of all charts of the form $(U_\beta \times \mathbb{R}^n, k_\beta)$, where (U_β, h_β) belongs to $\mathcal{B}$, determines an equivalent smooth atlas for $T(M)$.*

Proof. Since $T(\mathcal{B})$ is contained in the standard smooth atlas, which we shall call $T(\mathcal{A})$, it suffices to show that the sets $k_\beta(U_\beta \times \mathbb{R}^n)$ form an open covering of $T(M)$. The proposition regarding the map τ_M provides a quick way of verifying the open covering assertion. Since $\mathcal{B}$ is an atlas for M, the sets $h_\beta(U_\beta)$ form an open covering of M; consequently, their inverse images with respect to τ_M form an open covering of $T(M)$. However, in the proof of the proposition on τ_M we have shown that $\tau^{-1}(h_\beta(U_\beta))$ is equal to $k_\beta(U_\beta \times \mathbb{R}^n)$, and thus we have shown that sets of the latter type form an open covering of $T(M)$.■

III.5.3 : *Vector space operations in the tangent bundle*

Now that we have constructed the tangent bundle, we need to show that it can be viewed as a union of n-dimensional vector spaces, with one for each point in the manifold; as noted previously, we would like this family of vector space to be continuously parametrized by the points of the manifold in some reasonable sense that must be defined.

Notation. If $x \in M$ where M is a smooth manifold, then $T_x(M)$ is defined to be the inverse image $\tau_M^{-1}(\{x\})$; this subspace is homeomorphic to $\mathbb{R}^n$ by construction and is called the **tangent space to** x **in** M, or more generically the **fiber** or x with respect to the map τ_M.

Formally, here is the structure that we want:

BASIC OBJECTS:

1. A parametrized zero map; specifically, a smooth map $z : T(M) \to T(M)$ such that $\tau_M \circ z = \tau_M$.

2. A parametrized scalar multiplication map; specifically, a smooth map $\mu : \mathbb{R} \times T(M) \to T(M)$ such that $\tau_M \circ \mu = \tau_M \circ \pi_{T(M)}$, where $\pi_{T(M)}$ denotes projection onto the $T(M)$ factor.

3. A smooth structure defined on the space $T(M) \times_M T(M)$, which is the inverse image of the diagonal $\Delta_M \subset M \times M$ under the squared projection map $\tau_M \times \tau_M$. If $\tau_2(M)$ denotes either of the maps

$$q \circ (\tau_m \times \tau_M) | \, T(M) \times_M T(M)$$

where q denotes projection onto the first or second factor (these are equal by the definition of $T(M) \times_M T(M)$!), then $\tau_2(M)$ is to be smooth with respect to this smooth structure.

4. A parametrized vector addition map; specifically, a smooth map $\Sigma : T(M) \times_M T(M) \to T(M)$ such that $\tau_M \circ \Sigma = \tau_2(M)$.

If z, μ and Σ are mappings as above, then it follows that z maps $T_x(M)$ to itself, μ maps $\mathbb{R} \times T_x(M)$ to $T_x(M)$, and Σ maps the fiber of x with respect to $\tau_2(M)$ — which is $T_x(M) \times T_x(M)$ — to $T_x(M)$. We shall denote the associated maps of fibers by z_x, μ_x and Σ respectively. With this notation we can state the final thing that we need fairly simply.

BASIC PROPERTY OF THESE OBJECTS: ∞. For each $x \in M$ the maps z_x, μ_x and Σ define an n-dimensional real vector space structure on $T_x(M)$, with Σ defining the vector addition, z_x defining the zero vector, and μ_x defining the scalar multiplication.

If U is open in $\mathbb{R}^n$ then we can do this very directly on $T(U) = U \times \mathbb{R}^n$ by simply taking the standard vector space operations that each set $\{x\} \times \mathbb{R}^n$ inherits from $\mathbb{R}^n$ with its usual vector space operations. One can also define vector space structures for the tangent spaces to points in an n-dimensional level set $L \subset \mathbb{R}^{m+n}$; in this case the tangent space to a point x in L is essentially an n-dimensional vector subspace of $\{x\} \times \mathbb{R}^{m+n}$. We shall proceed by using the first of these as a model, and later we shall see that our construction yields the vector space operations on the tangent spaces of level sets that we have described.

CONSTRUCTION OF THE ZERO VECTOR MAP. We define the map locally using charts and then prove that the definitions for different charts are compatible. Given a chart $(U\alpha \times \mathbb{R}^n, k_\alpha)$ for $T(M)$, define $z_\alpha : U_\alpha \times \mathbb{R}^n \to T(M)$ by the formula

$$z_\alpha(x, \mathbf{v}) \quad = \quad k_\alpha(x, \mathbf{0}) \; .$$

In order to show this yields a well-defined map on the tangent space we need to check that $z_\alpha \circ \varphi_{\alpha\beta} = z_\beta$ when the left hand side is defined. This is true by the following sequence of equations:

$$z_\alpha \circ \varphi_{\alpha\beta}(x, \mathbf{v}) \quad = \quad z_\alpha(\psi_{\alpha\beta}(x), D\psi_{\beta\alpha}(x)\mathbf{v}) \quad = \quad k_\alpha(\psi_{\alpha\beta}(x), \mathbf{0}) \quad =$$

$$k_\alpha(\psi_{\alpha\beta}(x), D\psi_{\beta\alpha}(x)\mathbf{0}) \;=\; k_\alpha \circ \varphi_{\beta\alpha}(x, \mathbf{0}) \;=\; k_\beta(x, \mathbf{0}) \;=\; z_\beta(x, \mathbf{v})$$

CONSTRUCTION OF THE SCALAR MULTIPLICATION MAP. In this case the local definition is

$$\mu_\alpha(t, x, \mathbf{v}) \;=\; k_\alpha(x, t\mathbf{v})$$

and the compatibility of these maps is true by a similar sequence of equations:

$$\mu_\alpha \circ \left[\mathrm{id}_{\mathbb{R}} \times \varphi_{\alpha\beta}\right](t, x, \mathbf{v}) \;=\; \mu_\alpha(t,\, \psi_{\alpha\beta}(x),\, D\psi_{\beta\alpha}\mathbf{v}) \;=$$

$$k_\alpha(\psi_{\alpha\beta}(x),\, t \cdot D\psi_{\beta\alpha}\mathbf{v})) \;=\; k_\alpha(\psi_{\alpha\beta}(x),\, D\psi_{\beta\alpha}(t\mathbf{v})\,) \;=$$

$$k_\alpha \circ \varphi_{\alpha\beta}(x, t\mathbf{v}) \;=\; k_\beta(x, t\mathbf{v}) \;=\; \mu_\beta(t, x, \mathbf{v})$$

CONSTRUCTION OF A SMOOTH STRUCTURE ON $T(M) \times_M T(M)$. First of all, we note that each fiber $\tau_2(M)^{-1}(\mathrm{pt.}\}$ is homeomorphic to $\mathbb{R}^n \times \mathbb{R}^n$ and that the map $\tau_2(M)$ from $T(M) \times_M T(M)$ to $T(M)$ is continuous and open. One can prove that $T(M) \times_M T(M)$ is Hausdorff and second countable by the same sort of argument employed for $T(M)$; filling in the details is left to the reader as an exercise.

A smooth atlas may be defined as follows: Let $(U_\alpha \times \mathbb{R}^n, k_\alpha)$ be a coordinate chart for $T(M)$, note that the map

$$k_\alpha \times k_\alpha | \Delta_{U_\alpha} \times \mathbb{R}^n \times \mathbb{R}^n$$

is contained in $T(M) \times_M T(M)$, and define

$$\lambda_\alpha : U_\alpha \times \mathbb{R}^n \times \mathbb{R}^n \longrightarrow TM) \times_M T(M)$$

to be the map determined by $k_\alpha \times k_\alpha$ in this manner. This yields a smooth atlas because the transition maps

$$\Theta_{\beta\alpha} : V_{\beta\alpha} \times \mathbb{R}^n \times \mathbb{R}^n \longrightarrow V_{\alpha\beta} \times \mathbb{R}^n \times \mathbb{R}^n$$

are given by the formula

$$\Theta_{\beta\alpha}(x, \mathbf{v}, \mathbf{w}) \;=\; \left(\,\psi_{\beta\alpha}(x),\, D\psi_{\beta\alpha}(x)\mathbf{v},\, D\psi_{\beta\alpha}(x)\mathbf{w}\,\right) .$$

It follows from these definitions that the projection $\tau_2(M)$ is smooth.

CONSTRUCTION OF THE VECTOR ADDITION MAP. We now define addition by the formula

$$\Sigma_\alpha(x, \mathbf{v}, \mathbf{w}) \;=\; k_\alpha(x, \mathbf{v} + \mathbf{w}\,) .$$

Once again a lengthy computation is needed to prove the required consistency condition

$$\Sigma_\alpha\left(\,\Theta_{\alpha\beta}(x, \mathbf{v}, \mathbf{w})\,\right) \;=\; \Sigma_\beta(x, \mathbf{v}, \mathbf{w})$$

and as in the preceding two arguments the linearity of $D\psi_{\alpha\beta}(x)$ plays a crucial role in the verification. Details of this are left to the reader as an exercise.

In the preceding discussion we did not explicitly discuss the proofs of identities such as $\tau_m \circ z = \tau_m$ and the corresponding identities for μ and Σ. Once again it is left to the reader to verify that all these properties hold. Here is a hint in the case of the zero map: By construction the local maps z_α satisfy $\tau_M \circ z_\alpha(x, \mathbf{v}) = h_\alpha(x)$, and the same is also true for $\tau_M \circ k_\alpha(x, \mathbf{v})$.■

TANGENT BUNDLES FOR TOPOLOGICAL MANIFOLDS. (‡) Results of J. Milnor, J. M. Kister and B. Mazur from the nineteen sixties yield a partial generalization of the tangent bundle to arbitrary topological manifolds. More precisely, the topological tangent bundle for an n-manifold is a pair $(E, p : E \to M)$ such that E is a topological $2n$-manifold and p is a continuous map such that the following holds:

> Each $x \in M$ has an open neighborhood V such that V is an open subset of $\mathbb{R}^n$ and there is a homeomorphism $k : U \times \mathbb{R}^n \to p^{-1}(V)$ such that $p(k(x, y)) = x$ for all $(x, y) \in U \times \mathbb{R}^n$.

Further information on the construction of this object appears in the references cited below.

Note that there is no assumption about vector space operations on the fibers $p^{-1}(\{z\})$ where z runs through all the points of M. In fact, results from the previously cited book of Kirby and Siebenmann show that a manifold of dimension $\neq 4$ has a smooth structure if and only if one can impose reasonable continuously parametrized family of vector space structures on the fibers.

[x] J. M. Kister, *Microbundles are fibre bundles*, Ann. of Math. (2) **80** (1964), 190–199.

[x] J. W. Milnor, *Microbundles. I*, Topology **3** Suppl. 1 (1964), 53–80.

III.5.4 : *Naturality of the tangent bundle construction*

The aim of this subsection is to show that *the tangent space construction for smooth manifolds extends also yields a compatible construction for smooth maps of smooth manifolds.*

Here is a summary of the main construction:

THEOREM. *Let $f : M \to N$ be a smooth map of smooth manifolds (we suppress the atlases here to simplify the notation) where $\dim M = m$ and $\dim N = n$. Then there is a canonical smooth map $T(f) : T(M) \to T(N)$ such that the following hold:*

(i) For each $p \in M$, $T(f)$ sends $T_p(M)$ linearly to $T_{f(p)}(N)$.

(ii) If we have smooth charts (U_α, h_α) for M and (V_β, k_β) for N such that $f(h_\alpha(U_\alpha)) \subset k_\beta(V_\beta)$ and the maps for the associated charts in the tangent space atlases are denoted by H_α and K_β, then $T(f)$ maps $H_\alpha(U_\alpha \times \mathbb{R}^m)$ into $K_\beta(V_\beta \times \mathbb{R}^n)$ and "$K_\beta^{-1} \circ T(f) \circ H_\alpha$"$(x, \mathbf{v})$ is equal to ("$k_\beta^{-1} \circ f \circ h_\alpha$"$(x)$, D"$k_\beta^{-1} \circ f \circ h_\alpha$"$(x)\mathbf{v}$).

Proof. The second condition suggests that we define $T(f)$ on the image of a chart $H_\alpha(U_\alpha \times \mathbb{R}^n)$ by the given formula. This presupposes that f sends the image of h_α into the image of some chart for some atlas for N, but we know that we can find an atlas of charts for M with this property. If we let $f_1 : U_\alpha \to V_\beta$ be the local map determined by f — in other words, the map we have been describing as "$k_\beta^{-1} \circ f \circ h_\alpha$" most of the time — then we would like to say that

$$T(f) \circ H_\alpha(x, \mathbf{v}) \quad = \quad K_\beta(\, f_1(x),\, Df_1(x)\mathbf{v}\,) \; .$$

We need to show that this definition satisfies the basic consistency condition if we compare it with the corresponding formula for charts H_γ and K_δ (once again we assume that f sends the image of h_γ to the image of k_δ, and we denote the map corresponding to f by f_2. In terms of formulas, we need to show that if $(y, \mathbf{w}) = \varphi_{\gamma\alpha}(x, \mathbf{v})$, then

$$K_\beta\,(f_1(y),\, Df_1(y)\mathbf{w}) \quad = \quad K_\delta\,(f_2(x),\, Df_2(x)\mathbf{v}) \; .$$

The constructions of f_1 and f_2 from the original mapping f imply a consistency identity

$$f_1(\,\psi_{\gamma\alpha}(x)\,) \quad = \quad \psi_{\delta\beta}(\,f_2(x)\,)$$

whenever either side of the equation is defined. Direct calculation using this identity and the Chain Rule then yields the following sequence of equations:

$$K_\beta\,(f_1(y),\,Df_1(y)\mathbf{w}) \quad = \quad K_\beta\,(\,f_1(\psi_{\gamma\alpha}(x)),\,Df_1(\,\psi_{\gamma\alpha}(x)\,)\,[D\psi_{\gamma\alpha}(x)\mathbf{v}]\,) \quad =$$

$$K_\beta\,(\,f_1(\psi_{\gamma\alpha}(x)),\,D\,[f_1\circ\psi_{\gamma\alpha}(x)]\,\mathbf{v}) \quad = \quad K_\beta\,(\,\psi_{\beta\delta}(f_2(x)),\,D\,[\psi_{\delta\beta}\circ f_2(x)]\,\mathbf{v}) \quad =$$

$$K_\beta\,(\,\psi_{\delta\beta}(f_2(x)),\,D\psi_{\delta\beta}(f_2(x))\,[Df_2(x)\mathbf{v}]\,) \quad = \quad K_\beta\circ\varphi_{\delta\beta}\,(f_2(x),\,Df_2(x)\mathbf{v}\,)$$

By the defining construction for the tangent bundle, we know that the final expression is equal to $K_\delta\,(f_2(x),\,Df_2(x)\mathbf{v})$, and this completes the verification of the identity

$$K_\beta\,(f_1(y),\,Df_1(y)\mathbf{w}) \quad = \quad K_\delta\,(f_2(x),\,Df_2(x)\mathbf{v})$$

that we needed to conclude the existence of $T(f)$.■

In the language of category theory, the next result states that the constructions $M \longrightarrow T(M)$ and $f \longrightarrow T(f)$ define a **covariant functor** from the category of smooth manifolds to itself.

THEOREM. *The construction $f \to T(f)$ has the following properties:*

(*a*) $T(\mathrm{id}_M) = \mathrm{id}_{T(M)}$.

(*b*) *If $f : M \to N$ and $g : N \to P$ are smooth then $T(g\circ f) = T(g)\circ T(f)$.*

Proof. We shall first verify (*a*). — The definition of $T(\mathrm{id}_M)$ implies that if one takes a typical chart of the form $(U_\alpha \times \mathbb{R}^n, k_\alpha)$ then $T(\mathrm{id}_M)\circ k_\alpha(x,\mathbf{v}) = k_\alpha(x, D\mathrm{id}_\alpha(x)\mathbf{v})$, and since the derivative of an identity map is always the identity, it follows that the right hand side is equal to $k_\alpha(x,\mathbf{V})$. It follows that the restriction of $T(\mathrm{id}_M)$ to each open set of the form $k_\alpha(U_\alpha \times \mathbb{R}^n)$ is equal to the corresponding restriction of the identity map on $T(M)$. Since open sets of the form $k_\alpha(U_\alpha \times \mathbb{R}^n)$ form an open covering for $T(M)$, it follows that $T(\mathrm{id}_M)$ must be equal to $\mathrm{id}_{T(M)}$.

We shall now verify (*b*). — By construction, $T(f)$ and $T(g)$ are given as follows: First, one finds typical charts $(U_\alpha \times \mathbb{R}^n, k^M_\alpha)$, $(V_\beta \times \mathbb{R}^q, k^N_\beta)$, and $(W_\gamma \times \mathbb{R}^s, k^P_\gamma)$ for M, N and P respectively such that

(1) f maps the image of U_α in M to the image of V_β in N,

(2) g maps the image of V_β in N to the image of W_γ in P.

We shal denote the smooth maps from U_α to V_β and V_β to W_γ corresponding to f and g by f_1 and g_1 respectively. Then $T(f)$ and $T(g)$ are uniquely defined by the following identities:

$$T(f)\circ k^M_\alpha(x,\mathbf{v}) \quad = \quad k^N_\beta(\,f_1(x),\,Df_1(x)\mathbf{v}\,)$$

$$T(g)\circ k^N_\beta(y,\mathbf{w}) \quad = \quad k^P_\beta(\,g_1(y),\,Dg_1(y)\mathbf{w}\,)$$

Direct calculation then yields the following identity characterizing $T(g)\circ T(f)$:

$$T(g)\circ T(f)\circ k^M_\alpha(x,\mathbf{v}) \quad = \quad k^P_\gamma(\,g_1\circ f_1(x),\,[Dg_1(f_1(x))\cdot Df_1(x)]\,\mathbf{v}\,)$$

On the other hand, in the setting of the previous paragraph we also know that $g \circ f$ maps the image of U_α to the image of W_γ, and in fact the corresponding map from U_α to W_γ is just $g_1 \circ f_1$. Therefore the map $T(g \circ f)$ is uniquely defined by the following identity:

$$T(g \circ f) \circ k^M_\alpha(x, \mathbf{v}) \quad = \quad k^P_\gamma(\, g_1 \circ f_1(x),\, D\,[g_1 \circ f_1]\,(x)\mathbf{v}\,)$$

We now compare the final expressions in the two equations at the ends of the preceding paragraphs. The first coordinates are equal by construction, and the second are equal because the Chain Rule implies that

$$D\,[g_1 \circ f_1]\,(x) \quad = \quad [Dg_1(f_1(x)) \cdot Df_1(x)]$$

and therefore the restrictions of $T(g) \circ T(f)$ and $T(g \circ f)$ are equal on the set $k^M_\alpha(U_\alpha \times \mathbb{R}^n)$. Since these sets form an open covering for $T(M)$, it follows that $T(g \circ f) = T(g) \circ T(f)$ as required.■

Given a smooth map $f : M \to N$ and $p \in M$ it is often convenient to use $T_p(f)$ to denote the associated linear map from $T_p(M)$ to $T_{f(p)}(N)$.

III.5.5 : *Useful descriptions of some tangent spaces* $(\star)$

It is often useful to have simplified descriptions of tangent bundles when working with specific examples or abstract constructions. Here are some basic identities that arise fairly often in the subject.

THEOREM. *We have the following isomorphisms:*

(i) $T(\mathbb{R}^n) \cong \mathbb{R}^n \times \mathbb{R}^n$ *such that* τ *corresponds to projection onto the first factor and the vector space operations on* $\{pt.\} \times \mathbb{R}^n$ *are given by the standard* $1-1$ *correspondence between the latter and* $\mathbb{R}^n$*.*

(ii) If M *and* N *are smooth manifolds, then* $T(M \times N) \cong T(M) \times T(N)$ *such that* $\tau_{M \times N}$ *correspond to* $\tau_m \times \tau_N$*.*

(iii) If P *is a smooth manifold and* V *is an open subset of* P*, then* $T(V) \cong \tau_M^{-1}(V)$ *such that* τ_V *corresponds to* $\tau_M | T(V)$*.*

Note that the first and the third have the following consequence:

COROLLARY. *If* U *is an open subset of* $\mathbb{R}^n$*, then* $T(U) \cong U \times \mathbb{R}^n$ *such that* τ *corresponds to projection onto the first factor and the vector space operations on* $\{pt.\} \times \mathbb{R}^n$ *are given by the standard* $1-1$ *correspondence between the latter and* $\mathbb{R}^n$*.*■

Proofs of these identities are left to the exercises for this section.■

III.6 : Regular mappings and submanifolds

(Conlon, §§ 1.5, 2.5, 3.7)

This section has two related goals. The first is to formulate a general concept of smooth submanifold generalizing the two previously considered special cases: Open subsets of smooth