The centers of the matrix groups $U(n)$ and $SU(n)$

This note proves an assertion in the hints for one of the Additional Exercises for Chapter 7.

THEOREM. *Let $U(n)$ be the group of unitary $n \times n$ matrices (the entries are complex numbers, and the inverse is the conjugate of the transpose), and let $SU(n)$ be the kernel of the determinant homomorphism $U(n) \to \mathbb{C} - \{0\}$. Then the centers of both subgroups are the matrices of the form cI, where (as usual) I denotes the identity matrix and $|c| = 1$. In particular, the center of $SU(n)$ is a finite cyclic group of order n.*

The argument relies heavily on the Spectral Theorem, which implies that for every unitary matrix A there is a unitary matrix P such that PAP^{-1} is diagonal.

Proof. We shall first prove the result for $U(n)$. If A lies in the center then for each unitary matrix P we have $A = PAP^{-1}$. Since the Spectral Theorem implies that some matrix PAP^{-1} is diagonal, it follows that A must be diagonal. We claim that all the diagonal entries of A must be equal. Suppose that $a_{j,j} \neq a_{k,k}$. If P is the matrix formed by interchanging the j^{th} and k^{th} columns of the identity matrix, then P is a unitary matrix and $B = PAP^{-1}$ is a diagonal matrix with $a_{j,j} = b_{k,k}$ and $b_{j,j} = a_{k,k}$. But this means that A does not lie in the center of $U(n)$. Therefore the only matrices which can lie in the center of $U(n)$ have the form cI, and since we are working with unitary matrices it follows that $|c|$ must be 1.

We now turn to the case of $SU(n)$. If $D \subset U(n)$ is the subgroup of diagonal matrices, then of course D is central and we have $U(n) = D \cdot SU(n)$; in other words, every unitary matrix is the product of a matrix in $SU(n)$ and a diagonal matrix. Suppose now that $A \in SU(n)$ lies in the center of $SU(n)$; then the observations in the preceding sentence imply that A lies in the center of $U(n)$, so it follows that the center of $SU(n)$ is equal to $D \cap SU(n)$. Therefore the determination of the center reduces to specifying which diagonal matrices cI (where $|c| = 1$) have determinant equal to 1. But the determinant of cI is c^n, so the center consists of all matrices cI such that $c^n = 1$; *i.e.*, the center consists of all matrices cI such that c is a complex n^{th} root of 1. Since the set of all such matrices is isomorphic to $\mathbb{Z}_n$, we see that the center of $SU(n)$ is a cyclic group of order n.■