~

Mathematics 205C, Spring 2016, Examination 1

Answer Key

1. [*25 points*] Let M be a topological n-manifold.

(i) Explain why the connected components of M^n are open sets.

(ii) Explain why every point p of M has an open neighborhood base of open subsets U_α such that each punctured neighborhood $U_\alpha - \{p\}$ has at most two connected components.

SOLUTION

(i) This is true because every point in M has an open subset homeomorphic to an open subset of $\mathbb{R}^n$. Since every point in $\mathbb{R}^n$ has a neighborhood base of connected open subsets, the same is true for M and hence M is locally connected. Now the connected components of a locally connected space are open, and therefore the connected components of M must be open.$\blacksquare$

(ii) By the preceding, every point has a neighborhood base of open subsets U_α which are homeomorphic to a metric disk $N_h(x, \mathbb{R}^n)$. For each of these subsets, the punctured neighborhood $N_h(x, \mathbb{R}^n) - \{x\}$ is connected if $n \geq 2$ and has two components if $n = 1$.$\blacksquare$

2. [*20 points*] Suppose that M is a smooth manifold and $\Delta : M \to M \times M$ is the diagonal map $\Delta(x) = (x,x)$. Prove that Δ is a smooth map and in fact is a smooth immersion. [*Hint:* If p is projection onto either factor, what is $p\circ\Delta$ and what does this imply for the induced map of tangent spaces?]

SOLUTION

A map h into $M \times M$ is smooth if and only if its composites with the two coordinate projections $p_i : M \times M \to M$ is smooth. Since $p_1 \circ \Delta = \mathrm{id}_M = p_2 \circ \Delta$, the smoothness of Δ follows from the first sentence and the smoothness of the identity mapping.

The fact that Δ is an immersion follows because Δ is a retract; *i.e.*, we have

$$\mathrm{id}_{T(M)} \quad = \quad T(\mathrm{id}_M) \quad = \quad T(p_i \circ \Delta) \quad = \quad T(p_i)\circ T(\Delta) \ .$$

In particular, for each $x \in M$ this means that the composite linear mapping of tangent spaces $T_{\Delta(x)}(p_i)\circ T_x(\Delta)$ is the identity on $T_x(M)$. Therefore the linear mapping $T_x(\Delta)$ has a left inverse, and the latter implies that $T_x(\Delta)$ is $1-1$, so that Δ is an immersion.■

3. [*30 points*] Let U and$V$ be open in $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, let $x \in U$, and suppose that the derivative matrix $Df(x)$ has rank k. Prove that there is some open neighborhood $U_0 \subset U$ of x such that $Df(y)$ has rank $\geq k$ for all $y \in U_0$. Also give an example such that the rank of $Df(y)$ is strictly greater than k for all $y \in U_0$ such that $y \neq x$. [*Hint for the first part:* Recall that the rank can be described in terms of determinants of square submatrices. *Hint for the second part:* There are examples where $U = V = \mathbb{R}$.]

SOLUTION

An $m \times n$ matrix A has rank $\geq k$ if and only if there is a $k \times k$ submatrix B of A (formed by deleting suitable suitable rows and columns) which is invertible, or equivalently its determinant is nonzero.

We are given that the derivative matrix $Df(x)$ has rank k, so let $P(x)$ be a $k \times k$ submatrix of $Df(x)$ such that $\det P(x) \neq 0$. Since the determinant is a polynomial in the entries of a matrix, it follows that if $P(y)$ is the corresponding submatrix of $Df(y)$ for $y \in U$, then $\det P(y) \neq 0$ for all y in some open neighborhood U_0 of x. Another application of the statement in the first paragraph shows that $Df(y)$ has rank $\geq k$ for all $y \in U_0$.

For the last part, it suffices to give an example of a smooth function $f : \mathbb{R} \to \mathbb{R}$ such that $f'(0) = 0$ and $f'(x) \neq 0$ for $x \neq 0$. There are many choices, but probably the simplest one is $f(x) = x^2$.$\blacksquare$

4. [*25 points*] Let $W \subset \mathbb{R}^3$ be the set of all points (ρ, θ, ϕ) such that $\rho \neq 0$, and let $\mathbf{S} : W \to \mathbb{R}^3$ be the spherical coordinate transformation:

$$\mathbf{S}(\rho, \theta, \phi) \quad = \quad (\rho \cos\theta \sin\phi, \rho \sin\theta \sin\phi, \rho \cos\phi)$$

Determine whether $\mathbf{S}$ is a submersion. If not, find a subset $E \subset \mathbb{R}^3$ which is a union of surfaces such that the restriction of $\mathbf{S}$ to $W - E$ is a submersion. [*Hint:* Recall that a plane in $\mathbb{R}^3$ is a special type of surface. Also recall the hint from the preceding problem.]

SOLUTION

Since the map goes from an open subset of $\mathbb{R}^3$ into $\mathbb{R}^3$, we need to determine when $D\mathbf{S}$ has rank 3, and this is equivalent to the nonvanishing of the familiar Jacobian

$$\frac{\partial(x, y, z)}{\partial(\rho, \theta, \phi)} \quad = \quad \rho^2 \sin\phi$$

which one encounters in elementary multivariable calculus. This Jacobian vanishes if and only if ϕ is an integral multiple of π. Therefore, if F is the union of the parallel planes $\phi = n\pi$ (where $n \in \mathbb{Z}$) and E is the union of the open half-planes $\{\phi = n\pi, \rho \neq 0\}$, then it follows that the complement $W - E \subset \mathbb{R}^3$ is the unique maximal open subset of $\mathbb{R}^3$ such that $\mathbf{S}$ is a smooth submersion on $W - E$.■

FOOTNOTE. Usually the change of variables formula for multiple integrals has an assumption that the Jacobian is nowhere zero, and strictly speaking this means that the standard change of variables rule does not apply directly to spherical coordinates. Fortunately, one can work around this problem because the set of all points with vanishing Jacobian in (ρ, θ, ϕ)-space, and also its image in (x, y, z)-space, have measure zero, so it does not matter what happens on the subset where the Jacobian vanishes (one is integrating over sets of measure zero!). For this example one can check directly that the image has measure zero without using Sard's Theorem: The image is just the z-axis.■