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Mathematics 205C, Spring 2016, Examination 2

Answer Key

1. [*25 points*] Let M and N be connected smooth manifolds, let $p : \widetilde{N} \to N$ be a universal covering space projection where $\widetilde{N}$ has the smooth structure induced from N, let $f : M \to N$ be a smooth map, and assume that there is a continuous lifting $F : M \to \widetilde{N}$, so that $f = p \circ F$. Prove that the lifting F is also smooth.

SOLUTION

For each $x \in M$ we need to find an open neighborhood V of x such that $F|V$ is smooth.

Let W be a connected open neighborhood of $f(x)$ such that there is a coordinate chart $f : W \to \Omega$, where Ω is open in $\mathbb{R}^n$, with $n = \dim N$, and W is evenly covered with respect to the covering space projection p. Let W' be the sheet of $p^{-1}[W]$ containing $F(x)$. By continuity there is a connected open neighborhood V of x such that $f[V] \subset W$. By connectedness, F must map all of U into the sheet W'. Since W' maps diffeomorphically to W, it follows that $F|V$ is defined by the composite of $W|f|V : V \to W$ with the inverse to the diffeomorphism $W'|p|W$, and therefore $F|V$ is smooth.■

2. [*25 points*] Suppose that $f : M \to N$ and $g : N \to P$ are smooth immersions. Prove that the composite $g \circ f$ is also an immersion.

SOLUTION

Let $x \in M$. Since f and g are immersions, the linear maps of tangent spaces $T_x(f) : T_x(M) \to T_{f(x)}(N)$ and $T_{f(x)}(g) = T_{f(x)}(N) \to T_{gf(x)}(P)$ are both 1–1. Therefore their composite $T_x(g \circ f) = T_{f(x)}(g) \circ T_x(f)$ is 1–1, which means that $g \circ f$ is also an immersion.■

3. [*25 points*] Let $\omega \subset \mathbb{R}^n$ be the first orthant, which consists of all point $x = (x_1, \cdots, x_n)$ such that $x_i > 0$ for all i, and let $f : \Omega \to \mathbb{R}$ be the function $f(x) = x_1 \cdots x_n - 1$. show that the set of points $V \subset \Omega$ which satisfy $f(x) = 0$ is a smooth submanifold.

SOLUTION

We need to show that if $f(x) = 0$ then $Df(x) \neq 0$; the latter is equivalent to the nonvanishing of one of the partial derivatives

$$\frac{\partial f}{\partial y_i}(x) \ .$$

Now the displayed partial derivative is equal to $\prod_{j \neq i} x_j$, and since $x \in \Omega$ it follows that all of the factors in such a product is positive. In particular, if $x \in \Omega$ then all of the partial derivatives vanish, and therefore 0 is a regular value: If $f(x) = 0$ then $Df(x) \neq 0$.■

4. [*25 points*] Suppose that $f : M \to N$ is a smooth map of smooth manifolds and $\dim M < \dim N$. Explain why $N - f[M]$ is dense in N.

SOLUTION

Given $x \in N$, we need to show that every sufficiently small open neighborhood U of x contains at least one point which is not in $f[M]$. In this context, we shall interpret "sufficiently small" to mean that U is contained in some open set W such that there is a smooth chart $h : W \to \Omega$.

By the easy case of Sard's Theorem we know that $f[M]$ has measure zero. It follows that if $U \subset W$ then $h[U \cap f[M]] \subset h[U]$ has measure zero. Now a subset of measure zero in an open subset of $\mathbb{R}^n$ must have a dense complement (if the subset contained an open neighborhood of a point, then it would not have measure zero), and therefore there is some point $y \in h[U] - hf[M]$. If we choose z such that $h(z) = y$, then we have $z \in U - f[M]$, and as noted above this implies that $N - f[M]$ is dense in N.■

5. [*25 points*] If U is open in $\mathbb{R}^n$ and $h : U \to \mathbb{R}$ is smooth, then its gradient vector field is defined in the usual fashion as

$$\mathbf{X}_h \quad = \quad \sum_j \frac{\partial h}{\partial y_j} \frac{\partial}{x_j} \ .$$

Given two smooth functions f and g on U, compute the Lie bracket vector field $[\mathbf{X}_f, \mathbf{X}_g]$.

SOLUTION

Write $\partial_j = \frac{\partial}{\partial x_i}$. Since the Lie bracket is distributive with respect to addition, it follows that

$$[\mathbf{X}_f, \mathbf{X}_g] \quad = \quad \sum_{i,j} \left[\partial_i f \cdot \partial i, \partial_j g \cdot \partial j\right] \ .$$

We can now apply the identity $[aY, bZ] = a(Yb)Z - b(Za)Y + ab[X,Y]$ to each summand and use the vanishing of the brackets $[\partial_i, \partial_j]$ to expand and simplify each term in the sum. These considerations show that the right hand side is equal to

$$\sum_{i,j} \left(\partial_i f \cdot \partial_i \partial_j g\right) \partial_j \ - \ \left(\partial_j g \cdot \partial_j \partial_i f\right) \partial_i$$

and by interchanging the roles of i and j in the second terms we may rewrite this in the form

$$\sum_j \left(\sum_i \partial_i f \cdot \partial_i \partial_j g \ - \ \partial_i g \cdot \partial_i \partial_j f \right) \partial_j \ . \blacksquare$$

6. [*25 points*] Let $UT(3,\mathbb{R})$ be the set of all upper unitriangular matrices having the form

$$A = \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}$$

where $x, y, z \in \mathbb{R}$.

(i) Prove that $UT(3,\mathbb{R})$ is a Lie group with respect to ordinary matrix multiplication. [*Hint:* The inverse can be found either by solving the matrix equation $AX = I$, where X be assumed to be unitriangular, or by taking determinants of 2×2 submatrices.]

(ii) Let $A \in UT(3,\mathbb{R})$ be the matrix

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} .$$

Find a second matrix B such that $AB \neq BA$.

SOLUTION

(i) We shall take the obvious diffeomorphism from $UT(3;\mathbb{R})$ to $\mathbb{R}^3$ where the displayed matrix goes to (x, y, z). The smoothness of the multiplication map follows immediately from writing out the matrix product of two elements in $UT(3;\mathbb{R})$:

$$\begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & u & v \\ 0 & 1 & w \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x+u & v+xw+y \\ 0 & 1 & z+w \\ 0 & 0 & 1 \end{pmatrix}$$

If we set the right hand side equal to the identity matrix or use determinants, we can check that the inverse of the matrix with nontrivial entries (x, y, z) is the matrix with nontrivial entries $(-x, xz - y, -z)$, and therefore the inverse mapping on $UT(3;\mathbb{R})$ is also smooth.■

(ii) Let $B \in UT(3;\mathbb{R})$ be the matrix with nontrivial entries (x, y, z). Then the product AB is the matrix with nontrivial entries $(x+1, y+z, z)$, and the product BA is the matrix with nontrivial entries $(x + 1, y, z)$. Therefore $AB = BA$ if and only if $z = 0$, and accordingly $AB \neq BA$ if $z \neq 0$. Perhaps the simplest explicit example of a matrix B with $AB \neq BA$ is given by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} .■$$