EXERCISES FOR MATHEMATICS 205A

SPRING 2016 — Part 1

The headings denote chapters of the text for the course:

J. Lee, *Introduction to Smooth Manifolds* (Second Edition), Springer-Verlag, 2012.

Exercises which appear throughout the text are numbered in the form $m.n$, and exercises at the end of the chapters are numbered in the form $m - n$. Except when explicitly noted otherwise, it will suffice to prove exercises for manifolds without boundary.

A . Review of Mathematics 205A

Lee: A.22, A.23

Additional exercises

1. If X is a topological space, a family of subsets $\mathcal{E} = \{E_\alpha\}$ is said to be *locally finite* if each point $x \in X$ has an open neighborhood U_x such that U_x is disjoint from all but finitely many subsets in the family. Prove that if $\mathcal{E}$ is a locally finite family, then the closure of $\cup_\alpha E_\alpha$ is equal to the union $\cup_\alpha \overline{E_\alpha}$, where as usual $\overline{E}$ denotes the closure of E.

2. Let X be a topological space which is a union of an increasing countable sequence of subsets K_n such that for each n the set K_n is contained in the interior of K_{n+1}. Prove that every compact subset of X is contained in some K_n.

3. Explain why every open subset of some $\mathbb{R}^m$ satisfies the conditions in the previous exercise.

C . Review of multivariable calculus

Lee: C.5(c)

Additional exercises

1. Let U be an open subset of some $\mathbb{R}^q$, so that U is an increasing union of compact spaces K_n as in Additional Exercise A.1. Let $\{f_n\}$ be a sequence of smooth functions on U such that $m \geq n$ implies that $f_m|K_n = f_n|K_n$. Prove that the sequence $\{f_n\}$ converges uniformly on compact subsets and that the limit function is smooth. [*Hint:* If f is the limit function, why do we have $f|K_n = f_m|K_n$ for all $k \geq m$?]

2. Let A be an $m \times n$ matrix, and let $L_A : \mathbb{R}^n \to \mathbb{R}^m$ be the associated linear transformation $L_A(X) = AX$ (matrix product, viewing X as a column vector. Prove that there is some constant $K > 0$ such that the lengths of X and AX satisfy $|AX| \leq K \cdot |X|$ for all X. Also prove that if K is a compact subset and A is a continuous matrix valued function on K, then we can find some K such that $A(y)X \leq K|X|$ for all $y \in K$ and $X \in \mathbb{R}^n$.

3. Let U be an open disk in $\mathbb{R}^n$, and let $f : U \to \mathbb{R}^m$ be a smooth function (continuous partial derivatives of all orders for all coordinates). If C is a convex compact subset of U, prove that there is a constant K such that $|f(y) - f(x)| \leq K \cdot |y - x|$ for all $y, x \in K$. [*Hint:* Write $g(t) = f(ty + (1-t)x)$ for $t \in [0,1]$; this makes sense because K is convex. Now apply the Fundamental Theorem of Calculus and the preceding exercise.]

1 . Smooth manifolds

Lee, 1.17, 1.18, 1.20, $1-2$, $1-4$, $1-6$, $1-12$ (for *topological* manifolds only)

Additional exercises

1. Let X be a Hausdorff space, and suppose that X has an open covering $\{U_\alpha\}$ such that each U_α is a topological n-manifold for some fixed n. Prove that X is a topological manifold. Explain why this statement becomes false if the Hausdorff condition is removed.

2. A compact Hausdorff space Γ is a *graph* if it is a finite union of subspaces E_j such that each E_j is homeomorphic to the closed unit interval $[0,1]$ and if $i \neq j$ then $E_i \cap E_j$ is an endpoint of both E_i and E_j (note that one can characterize the endpoints topologically as the two points whose complements are connected). The set of endpoints of the subsets E_k is called the set of vertices of Γ and each E_k is called an edge of Γ. Prove that if Γ is a topological manifold, then every vertex lies on exactly two edges. [*Hint:* Look at the proof that the figure 8 curve is not a topological manifold.]

3. In the notation of the previous exercise, it follows that if one removes a finite set of points from Γ, then Γ is a topological 1-manifold. All letters in the alphabet and all Hindu-Arabic numerals admit decompositions into closed subspaces which make them into graphs. Assuming that the letters and numerals are given in the sans-serif form

A B C D E F G H I J K L M

N O P Q R S T U V W X Y Z

1 2 3 4 5 6 7 8 9 0

determine the least numbers of points that must be removed in order to obtain a topological 1-manifold.

4. Suppose that we are given two smooth n-manifolds M and N such that there is a diffeomorphism Φ from an open subset $U \subset M$ to an open subset $V \subset N$. Then we can form the quotient space $P = M \cup_\Phi N$, which is given by $M \amalg N$ modulo the equivalence relation generated by $x \equiv \Phi(x)$ for all $x \in U$. Assume that this space P is Hausdorff.

If $\mathcal{A}$ and $\mathcal{B}$ are maximal smooth atlases for M and N, let $\mathcal{F}$ be the corresponding family of local charts for P corresponding to $\mathcal{A}$ and $\mathcal{B}$. Prove that $\mathcal{F}$ is a smooth atlas for P. [*Hint:* The main point is to show that a chart in $\mathcal{A}$ is compatible with one in $\mathcal{B}$ and vice versa. Some way or other the diffeomorphism Φ should be relevant to this issue.]

5. Suppose that M is a topological m-manifold without boundary and N is a topological n-manifold with boundary (in the sense of the definition in the lectures!). Prove that the product

$M \times N$ is a topological $(m+n)$-manifold with boundary (in the sense of the definition in the lectures) and its boundary is $M \times \partial N$. Explain why the same is true if "topological" is replaced by "smooth" in the preceding discussion.

6. Let ε fun through $\{-1, 1\}$. Prove that the inverses to the mappings $h_{k,\varepsilon} : N_1(0, \mathbb{R}^3) \to S^3$ defined by

$$h_{1,\varepsilon}(x,y,z) = \left(\varepsilon\sqrt{1-x^2-y^2-z^2}, x, y, z\right), \qquad h_{2,\varepsilon} = \left(x, \varepsilon\sqrt{1-x^2-y^2-z^2}, y, z\right)$$

$$h_{3,\varepsilon}(x,y,z) = \left(x, y, \varepsilon\sqrt{1-x^2-y^2-z^2}, z\right), \qquad h_{4,\varepsilon} = \left(x, y, z, \varepsilon\sqrt{1-x^2-y^2-z^2}\right)$$

form a smooth atlas for $\mathbb{R}^3$. Formulate a more general result of this type to describe a smooth atlas for S^n with $2n+2$ charts.

7. Let $\mathcal{U}$ be an open covering of a smooth manifold M with maximal atlas $\mathcal{A}$. An smooth atlas $\mathcal{A}'$ is said to be $\mathcal{U}$-small if the domain of every smooth chart in $\mathcal{A}'$ is contained in some element of $\mathcal{U}$. Prove that $\mathcal{A}$ contains a $\mathcal{U}$-small atlas. [*Hint:* What can we say about the restriction of a chart in $\mathcal{A}$ to an open subset?]

2 : Smooth maps

Lee, 2.7, 2.16, $2-10$ ("if" direction only for part (c))

Additional exercises

1. Let M be a smooth manifold and let $f, g : M \to \mathbb{R}$ be smooth functions. prove that $f+g$ and fg are also smooth functions. Also, if f is never zero prove that $1/f$ is a smooth function. State and prove similar results for vector operations on vector valued functions, including the cross product on $\mathbb{R}^3$. [*Hint:* You may assume these are known if M is an open subset in some $\mathbb{R}^n$.]

2. As noted in the lectures, there are two definitions of a smooth homotopy between two smooth maps $f, g : M \to N$.

(1) For some $\varepsilon > 0$ there is a smooth map $h : M \times (-\varepsilon, 1+\varepsilon) \to N$ such that $h|M \times \{0\} = f$ and $h|M \times \{1\} = g$.

(2) For some $\varepsilon > 0$ there is a smooth map $h : M \times (0,1) \to N$ such that $h|M \times \{t\} = f$ for $t < \varepsilon$ and $h|M \times \{t\} = g$ for $t > 1-\varepsilon$.

Prove that these definitions are equivalent.

3. Suppose that $f : M \to M'$ and $g : N \to N'$ are smooth maps of smooth manifolds. Prove that the product map $f \times g$ is also a smooth map if $M \times N$ and $M' \times N'$ are given the product smooth structures. Furthermore, prove that if f and g are diffeomorphisms then so is $f \times g$. [*Hint:* A map into a product is smooth if and only if its coordinate projections are smooth.]

4. Suppose that M is a smooth manifold. Prove that the following maps are diffeomorphisms:

(a) The twist map from $M \times M$ to itself which sends (x,y) to (y,x).

(b) The cyclic coordinate permuation map from $M \times M \times M$ to itself which sends (x,y,z) to (y,z,x).

(c) The middle four shuffle from $M \times M \times M \times M$ to itself which sends $(x,y,z.w)$ to (x,z,y,w).

[*Hints:* Recall the hint in the previous exercise, and observe that if f is any of the maps given above, then some iterated composite of f with itself is the identity. Why does this imply that some other iterated composite of f with itself is an inverse?]

5. Let $f : M \to N$ be a smooth map of smooth manifolds, let $p : E \to N$ be a covering space projection, take the smooth structure on E induced by p, and let $F : M \to E$ be a continuous lifting of f. Prove that F is also smooth.

6. Let $p : \mathbb{R} \to S^1$ be the usual covering space projection $p(t) = (\cos t, \sin t)$, let M be a smooth manifold, let $\gamma : \mathbb{R} \to M$ be a periodic smooth curve satisfying $\gamma(t+n) = \gamma(t)$ for all real numbers t and integers n, and let $\gamma^* : S^1 \to M$ be the unique continuous function such that $p \circ \gamma^* = \gamma$. Prove that γ^* is also smooth.