# EXERCISES FOR MATHEMATICS 205A

# SPRING 2016 — Part 2

The headings denote chapters of the text for the course:

J. Lee, *Introduction to Smooth Manifolds* (Second Edition), Springer-Verlag, 2012.

Exercises which appear throughout the text are numbered in the form $m.n$, and exercises at the end of the chapters are numbered in the form $m - n$. Except when explicitly noted otherwise, it will suffice to prove exercises for manifolds without boundary.

## 3 . Tangent spaces

RECOMMENDATION. Use the approach to constructing the tangent space described in the lectures and the document `amalgamation.pdf`. However, note that coordinate charts in that document are defined as maps from open subsets in $\mathbb{R}^n$ into a manifold $M$, while in Lee and this course we have defined coordinate charts as maps from open sets in $M$ to open sets in $\mathbb{R}^n$. The relationship between the two formulations is that the maps in Lee are inverses to the maps in `amalgamation.pdf`.

Lee, $3-3$, $3-4$, $3-5$

*Additional exercises*

**0.** Prove the following statements; each is an immediate consequence of previous exercises.

($a$) If $M$ is a smooth submanifold of $N$ and $N$ is a smooth submanifold of $P$,then $M$ is a smooth submanifold of $P$.

($b$) If $P$ is a smooth submanifold of $M$ and $Q$ is a smooth submanifold of $N$, then $P \times Q$ is a smooth submanifold of $M \times N$.

**1.** Suppose that $f : M \to N$ is a a smooth homeomorphism. Prove that $f$ is a diffeomorphism if and only if $T(f)$ is 1–1 and onto.

**2.** One can construct the Klein bottle $KB$ using two smooth charts $(U_i, h_i)$ for $i = 1, 2$ where $U_1 = U_2 = \mathbb{R}^2 - \{0\}$ such that the overlapping images are given by $V_{21} = V_{12} = \{\ z \mid |z| < \frac{1}{2}$ or $|z| > 2\ \}$ and the transition diffeomorphisms $\psi_{ij}$ are both given by $\psi_{ij}(x, y) = (x, y)$ if $\sqrt{x^2 + y^2} > 2$ and $\psi_{ij}(x, y) = (x, -y)$ if $\sqrt{x^2 + y^2} < \frac{1}{2}$ What are the domains of the charts for the corresponding smooth atlas of the tangent space $T(KB)$, and what is the corresponding transition map for these charts?

**3.** A smooth curve $\gamma$ from an open interval $(a, b)$ to a smooth manifold $M$ is said to be regular if its tangent vector at every point is nonzero, and a continuous curve $\gamma$ from $(a, b)$ to $M$ is said to be *regularly piecewise smooth* if one can find a partition

$$a = s_0 < s_1 < \cdots < s_p = b$$

such that the restrictions of $\gamma$ to the pieces $(s_0, s_1], [s_1, s_2] \ \cdots \ [s_{p-1}, s_p)$ all extend to regular smooth curves on open intervals containing the pieces.

($i$) Prove that if $M$ is connected, then every pair of points can be joined by a regular piecewise smooth curve $(-\varepsilon, 1+\varepsilon)$; *i.e.*, for each $x, y \in M$ one can find such a curve $\gamma$ so that $\gamma(0) = x$ and $\gamma(1) = y$.

($ii$) Prove the following strengthening ot ($i$): Every pair of points can be joined by a regular smooth curve. [*Hint:* start with the conclusion of ($i$) and use the construction on the last three pages of `nicecurves.pdf` to smooth out the corner points.]

**4.** Assume we are in the setting of Additional Exercise I.4, where we are given two smooth $n$-manifolds $M$ and $N$ such that there is a diffeomorphism $\Phi$ from an open subset $U \subset M$ to an open subset $V \subset N$. Then we can form the quotient space $P = M \cup_\Phi N$, which is given by $M \amalg N$ modulo the equivalence relation generated by $x \equiv \Phi(x)$ for all $x \in U$. If this space $P$ is Hausdorff, then the cited exercise yields a smooth structure on $P$ which contains open subsets diffeomorphic to $M$ and $N$.

Prove that the space $T(M) \cup_{T(\Phi)} T(N)$ is Hausdorff, and with the smooth structure on it given in the earlier exercise it is diffeomorphic to $T(P)$.

**5.** Let $M$ be a smooth manifold, let $p : E \to M$ be a Hausdorff covering space projection, and take the smooth structure on $E$ given in the lectures. Prove that $T(p) : T(E) \to T(M)$ is also a smooth covering space projection of the same type, and if $h : M \to M$ is a covering space (deck) transformation then so is $T(h)$.

## 4 : Mersions and embeddings

CONVENTION. The word *mersion* refers to a map which is either an immersion or a submersion. There is also a related concept of $k$-mersion, which is a smooth mapping $f : M \to N$ such that for each $x \in M$ the tangent space mapping $T(f)_x : T_x(M) \to T_{f(x)}(N)$ has constant rank $k$.

Lee, 4.10, 4.38, $4-6$, $4-8$, $4-12$, $4-13$

*Additional exercises*

**1.** ($i$) Prove that the composite of two smooth immersions is a smooth immerson.

($ii$) Prove that the composite of two smooth submersions is a smooth submerson.

($iii$) Prove that the composite of two smooth embeddings is a smooth embedding.

**2.** Let $f : M \to M'$ and $g : N \to N'$ be smooth mappings.

($i$) Prove that if $f$ and $g$ are immersions then so is $f \times g : M \times N \to M' \times N'$.

($ii$) Prove that if $f$ and $g$ are submersions then so is $f \times g : M \times N \to M' \times N'$.

($iii$) Prove that if $f$ and $g$ are smooth embeddings then so is $f \times g : M \times N \to M' \times N'$.

**3.** Let $X$ be the $y$-axis in the Cartesian plane, and let $Y$ be the graph of $\sin \frac{1}{x}$ for $x > 0$. Prove that the map $X \amalg Y \to \mathbb{R}^2$ is an immersion but not an embedding; also show that the restrictions to the two pieces are embeddings.

**4.** Prove that there is no immersion from a compact $n$-manifold into $\mathbb{R}^n$.

**5.** Prove that there is no submersion from a compact $n$-manifold into $\mathbb{R}$. [*Hint:* Such a map attains a maximum value. What does this mean if we look at a smooth chart at a point where the maximum is attained?]

*Note.* If $M$ is connected and noncompact, then one can always construct a smooth submersion $M \to \mathbb{R}$.

**6.** Given an immersion from a 1-connected compact smooth manifold to a smooth manifold of the same dimension, prove that it is a covering space projection. Does the statement remain true if the manifolds are not necessarily compact? Prove this or give a counterexample.

**7.** Let $U$ and $V$ be open in $\mathbb{R}^n$, let $f : U \to V$ be a smooth *surjective* immersion/submersion, and suppose that $g : V \to \mathbb{R}^q$ is a continuous map such that $g \circ f$ is smooth. Prove that $g$ is also smooth.

**8.** A continuous map $f : A \to X$ is a *retract* if there is a continuous map $g : X \to A$ such that $g \circ f = \mathrm{id}_A$. Suppose that $A$ and $X$ smooth manifolds and $f$ and $g$ are smooth. Prove that $f$ is an immersion.

**9.** Suppose that $M$ is a noncompact smooth manifold and there is a smooth 1–1 immersion $f : M \to \mathbb{R}^N$. Prove that there is a smooth embedding $g : M \to \mathbb{R}^{N+1}$ such that $g[M]$ is a closed subset. [*Hint:* Recall that there is a proper map from $M$ to $\mathbb{R}$.]

**10.** ($i$) Suppose that $M$ and $N$ are smooth manifolds. A smooth map $f : M \to N$ is said to be a *retract* if there is a smooth map $g : N \to M$ such that $g \circ f = \mathrm{id}_M$. Prove that a smooth retract is a smooth immersion.

($ii$) A smooth map of smooth manifolds $r : N \to M$ is said to be a smooth *retraction* if there is a smooth map $j : M \to N$ such that $r \circ j = \mathrm{id}_M$. Prove that if $r$ is a retraction, then the restriction of $r$ to some neighborhood of $j(M)$ is a submersion.

($iii$) A continuous map of topological spaces is said to be a continuous retract if it satisfies the condition in ($i$). Prove that if $A$ and $X$ are Hausdorff then $j$ is a closed mapping. Why does this imply that $j$ maps $A$ homeomorphically onto its image? [*Hint:* To see that $A$ is closed, show that it is the set of all points such that $x = j \circ r(x)$.]

**11.** Let $z : M \to T(M)$ be the map which sends each point $x \in M$ to the zero vector in the tangent space $T_x(M)$. Prove that $z$ is a smooth embedding. [*Hint:* What does $z$ look like in local coordinates, and why is $\tau_m \circ z$ the identity?]

**12.** Let $z : M \to T(M)$ be given as in the previous exercise. Prove that $z[M]$ is a strong deformation retract of $T(M)$ and $\tau_M$ is an associated deformation retraction.

**13.** Let $n_1, \cdots, n_k$ be positive integers and let $N$ be their sum. Prove that there is a smooth embedding of $\prod_j S^{n_j}$ into $S^{N+1}$. [*Hint:* One always has smooth embeddings of $S^p \times \mathbb{R}^q$ in $\mathbb{R}^{p+q}$ and embeddings of $S^{q-1} \times \mathbb{R}$ in $\mathbb{R}^q$. Use these as part of an inductive argument.]

**14.** Suppose we have smooth maps $i : M \to N$ and $j : N \to L$ such that $j \circ i$ is a smooth embedding. Prove that $i$ is a smooth embedding.

**15.** Suppose that $M$ is a connected smooth manifold and $f : M \to N$ is a smooth immersion. Prove that the diagonal $\Delta_M$ is a connected component of $(f \times f)^{-1}[\Delta_N]$. [*Hint:* Recall that $f$ is locally 1–1.]

## 5 : Smooth submanifolds

Lee, $5-1$, $5-3$, $5-6$, $5-7$, $5-10$, $5-11$, $5-17$, $5-19$ (but disregard the last sentence)

*Additional exercises*

**1.** Suppose that $U$ is open in $\mathbb{R}^n$ and that $f : U \to \mathbb{R}^n$ and $g : U \to \mathbb{R}^m$ are smooth functions where $m < n$. Let $x \in U$ be a point on the level set $L$ on which $g(x) = 0$, and suppose that $Dg(x)$ has rank $m$ ($\Longrightarrow$ if we restrict to a suitable open neighborhood $V$ of $x$ in $U$, the set $L \cap V$ is a smooth submanifold of dimension $n-m$).

($i$) Suppose that $f|L$ has a local maximum at $x$. Prove that $\nabla f(x)$ is perpendicular to the tangent space $T_x(L)$. [*Hint:* What can we say about $D[f|L](x)$ under the given hypothesis?]

($ii$) If the coordinates of $g$ are given by $g_j$ (where $1 \leq j \leq m$), explain why the orthogonal complement of $T_x(L)$ is spanned by the vectors $\nabla g_j(x)$.

($iii$) Using the preceding parts of this exercise, derive the **Lagrange Multiplier Rule**: One can find $m$ constants (or Lagrange multipliers) $\lambda_j$ such that $\nabla f(x) = -\sum_j \nabla g_j(x)$ or equivalently $x$ (and the $\lambda_j$'s) determine a solution to the following system of equations:

$$\nabla \left( f + \sum_j \lambda_j g_j \right) = \mathbf{0} \qquad \text{AND} \qquad g(x) = 0$$

Note that this is a system of $m+n$ scalar equations in the $n$ coordinates of $x$ and the $m$ multipliers $\lambda_j$.

**2.** Let $Q \subset \mathbb{R}^{n+1}$ be the unit cube consisting of all $(x_0, \cdots x_n)$ such that $\max_i |x_i| = 1$. Prove that $Q$ is homeomorphic to $S^n$, that $Q$ has a smooth atlas for which $Q$ is diffeomorphic to $S^n$, but $Q$ is not a smooth submanifold of $\mathbb{R}^{n+1}$.

**3.** Let $X$ be the $y$-axis in the Cartesian plane, and let $Y$ be the graph of $\sin \frac{1}{x}$ for $x > 0$. Prove that $X \cup Y$ is an immersed but not embedded submanifold but that each of $X$ and $Y$ taken separately is an embedded submanifold.

**4.** Let $A$ be a real nonsingular symmetric $n \times n$ matrix and let $c$ be a nonzero real number. Show that the quadric hypersurface defined by the equation $\langle Ax,\ x \rangle = c$ is a smooth $(n-1)$-dimensional submanifold of $\mathbb{R}^n$.

**5.** Let $M$ be a noncompact smooth manifold. Prove that there is a smooth embedding $f : (-\varepsilon, \infty) \to M$ such that the image of $[0, \infty)$ is a closed subset.

**6.** ($i$) Let $V$ the set of points $(x, y, z) \in \mathbb{R}^3$ satisfying the equations $x + y + z = 0$ and $xyz = 2$, and let $A$ be the three point set consisting of $(2, -1, -1)$, $(-1, 2, -1)$ and $(-1, -1, 2)$. Prove that $V - A$ is a smooth submanifold of $\mathbb{R}^3$.

($ii$) Let $V$ be the set of points $(x, y, z) \in \mathbb{R}^3$ satisfying the equations $x^2 + 2xz - 2yz + z^2 = 0$ and $2x - y + z = 3$. Prove that $V$ is a smooth submanifold of $\mathbb{R}^3$.

**7.** Show that it is possible to make the subset of the plane defined by the equation $x^3 - y^2 = 0$ into a smooth manifold but that the set in question is not a smooth submanifold of $\mathbb{R}^2$. What happens for the set $x^4 - y^2 = 0$?

**8.** Let $f : S^2 \to \mathbb{R}^4$ be the smooth map sending $(x, y, z)$ to $(x^2 - y^2, xy, xz, yz)$. Show that $f(x, y, z) = f(-x, -y, -z)$ for all $(x, y, z)$ and that the associated map $g : \mathbb{RP}^2 \to \mathbb{R}^4$ on the quotient manifold is a smooth embedding.

**9.** Let $A \subset \mathbb{R}^2$ be the graph of the function $f(t) = |t|$. Prove that $A$ is a topologically locally flat submanifold of $\mathbb{R}^2$ but not a smooth submanifold. [*Hints:* Construct a homeomorphism from $\mathbb{R}^2$ to itself that sends $A$ to the $x$-axis. To show $A$ is not a smooth submanifold, derive a contradiction by finding two candidates for the tangent space at the origin.]

**10.** Consider the set $LF_{n,k}$ of *labeled flexible $n$-gons* in $\mathbb{R}^k$. These are the figures obtained by joining $n > 2$ straight line segments of unit length into a closed curve.

($i$) Suppose that $n$ is odd and $k = 2$. Prove that $LF_{n,2}$ is a smooth submanifold of $\mathbb{R}^2 \times T^{n-1}$ whose dimension is equal to $n$.

($ii$) Prove that the set of all such objects with no self-intersections is a smooth manifold.

*Exercises involving real projective spaces*

We shall take the construction in Example 1.5 of Lee's book (see p. 6) as the basic definition for the $n$-dimensional real projective space $\mathbb{RP}^n$. As indicated in the hint for Exercise 1.7 on p. 7, if the quotient map $\mathbb{R}^{n+1} - \{\mathbf{0}\} \to \mathbb{RP}^n$ is denoted by $\pi$, then $\pi|S^n$ is surjective, and in fact it follows that we can view $\mathbb{RP}^n$ as the quotient of $S^n$ by the equivalence relation $x \equiv y \Longleftrightarrow x = \pm\, y$. If we view $\mathbb{RP}^n$ in this way, there is a natural atlas given in terms of Examples 1.4 and 1.5 as follows: Let $\Omega_i \subset \mathbb{R}^{n+1} - \{\mathbf{0}\}$ be the set of points where the $i^{\text{th}}$ coordinate is nonzero (this is called $\widetilde{U_i}$ in Lee), and let $U_i^+ \subset S^n$ and $U_i \subset \mathbb{RP}^n$ be the open subsets described on p. 5 of Lee. It follows that the images of these two sets in $\mathbb{RP}^n$ are equal to the same open subset, and it also follows that the charts given by the inverses to the mappings

$$N_1(\mathbf{0}, \mathbb{R}^n) \longrightarrow U_i^+ \longrightarrow U_i \longrightarrow \mathbb{RP}^n$$

form a compatible smooth atlas. The verification of this is straightforward (the user should be able to do this!), but it is a little messy.

Note that $S^n - \Omega_i$ is equal to $U_i^+ \cup U_i^-$ (where the two pieces are disjoint) and that the antipodal map $T : S^n \to S^n$ sending $x$ to $-x$ sends each of these subsets to the other.

**11.** ($i$) Explain why the antipodal map is a diffeomorphism fropm $S^2$ to itself, and if $\pi : S^n \to \mathbb{RP}^n$ is the quotient map then $\pi \circ T = \pi$. [*Hint:* $T^2$ = identity implies that $T = T^{-1}$.]

($ii$) Let $Q$ be a smooth manifold. Using the preceding material and discussion, explain why a map $f : \mathbb{RP}^n \to Q$ is smooth if and only if $f \circ \pi$ is smooth.

**12.** Let $f : S^2 \to \mathbb{R}^4$ be the smooth map sending $(x, y, z)$ to $(x^2 - y^2, xy, xz, yz)$. Show that $f(x, y, z) = f(-x, -y, -z)$ for all $(x, y, z)$ and that the associated map $g : \mathbb{RP}^2 \to \mathbb{R}^4$ on the quotient manifold is a smooth embedding. [*Note:* There is no smooth embedding $\mathbb{RP}^2 \to \mathbb{R}^3$.]

**13.** Let $p(x_1, \cdots, x_{n+1})$ be a homogeneous polynomial of degree $d$ with real coefficients, so that we have the following identity:

$$p(cx_1, \cdots, cx_{n+1}) = c^d \cdot p(x_1, \cdots, x_{n+1}), \quad \text{for all} \quad c, x_j$$

($i$) Suppose that $(x_1, \cdots, x_{n+1})$ and $(y_1, \cdots, y_{n+1})$ determine the same point of $\mathbb{RP}^n$. Explain why $p(x_1, \cdots, x_{n+1}) = 0$ if and only if $p(y_1, \cdots, y_{n+1}) = 0$, so that it is meaningful to discuss the projective zero set $\mathbb{P}V(p)$ of $p$ in $\mathbb{RP}^n$.

($ii$) Let $V(p)$ be the zero set of $p$ in $\mathbb{R}^{n+1}$, so that the inverse image of $\mathbb{P}V(p)$ is $V(p) - \{\mathbf{0}\}$. Prove that if the derivative $Dp$ is nonzero at every point of $V(p)$, then $\mathbb{P}V(p)$ is a smooth compact $(n-1)$-dimensional smooth submanifold of $\mathbb{RP}^n$. [*Hints:* Why is $\mathbb{P}V(p)$ a closed subset, and why does it suffice to show that the intersection with each $U_i$ is a smooth submanifold?]

**14.** Determine whether the following homogeneous polynomial equations define smooth submanifolds of $\mathbb{RP}^n$:

($i$) $x_1x_2x_3 - x_4^3 = 0$, where $n = 3$.

($ii$) $x_1^2 + x_2^2 + x_3^2 - x_4^2 = 0$, where $n = 3$.

($iii$) $x_1^d + x_2^d + x_3^d + x_4^d = 0$, where $d > 1$ is an odd integer and $n = 3$.

($iv$) $x_1^3 - x_1x_3^2 - x_2^2x_3 = 0$, where $n = 2$.

[*Hint:* It might be helpful to look over each $U_i$ separately and restrict to the sets where the $i^{\text{th}}$ homogeneous coordinate is equal to 1.]

## 6 : Approximation and embedding theorems

Lee, $6-1$, $6-3$, $6-16abce$

*Additional exercises*

**1.** For each of the statements below, either prove that it is true or give a counterexample to show it is false.

($a$) If $M$ is a smooth manifold and $A \subset M$ is a set of measure zero, then the closure $\overline{A} \subset M$ also has measure zero.

($b$) If $M$ is a smooth manifold with $B \subset A \subset M$ and $A$ has measure zero, then $B$ also has measure zero.

**2.** Sard's Theorem implies that if $f : M \to N$ is smooth and $\dim M < \dim N$, then $f[M] \subset N$ has measure zero. This exercise considers the possibilities if the dimensions are equal.

($i$) Show that there are examples of smooth maps $f : M \to N$ such that $\dim M = \dim N$ and $f[M] \subset N$ has measure zero.

($ii$) Suppose that $f : M \to N$ is smooth such that $\dim M = \dim N$. Prove that $f[M] \subset N$ has measure zero if and only if for each $x \in M$, the linear map of tangent spaces $T_x(f) : T_x(M) \to T_{f(x)}(N)$ is not invertible.

**3.** ($i$) Suppose that $\dim M < n$, where $M$ is a compact smooth manifold, and let $f : M \to S^n$ be continuous. Prove that $f$ is homotopic to a constant map. [*Hints:* First approximate $f$ by a smooth map $g$ which is homotopic to $f$; this can be done by choosing $\varepsilon > 0$ sufficiently small. Use Sard's Theorem to show that $f$ is not onto, so that the image lies in $s^n - \{\mathbf{p}\}$ for some $\mathbf{p}$. Why does this imply that $g$ is homotopic to a constant?]

($ii$) Give an example to show that a similar conclusion does not hold if we replace $S^n$ with some other compact smooth $n$-manifold (*i.e.*, find $N^n$ and $M$ so that $\dim M < n$ and find a map $f : M \to N$ which is not homotopic to a constant).