EXERCISES FOR MATHEMATICS 205A

SPRING 2016 — Part 2

The headings denote chapters of the text for the course:

J. Lee, *Introduction to Smooth Manifolds* (Second Edition), Springer-Verlag, 2012.

Exercises which appear throughout the text are numbered in the form $m.n$, and exercises at the end of the chapters are numbered in the form $m - n$. Except when explicitly noted otherwise, it will suffice to prove exercises for manifolds without boundary.

3 . Tangent spaces

RECOMMENDATION. Use the approach to constructing the tangent space described in the lectures and the document `amalgamation.pdf`. However, note that coordinate charts in that document are defined as maps from open subsets in $\mathbb{R}^n$ into a manifold M, while in Lee and this course we have defined coordinate charts as maps from open sets in M to open sets in $\mathbb{R}^n$. The relationship between the two formulations is that the maps in Lee are inverses to the maps in `amalgamation.pdf`.

Lee, $3-3$, $3-4$, $3-5$

Additional exercises

1. Suppose that $f : M \to N$ is a a smooth homeomorphism. Prove that f is a diffeomorphism if and only if $T(f)$ is 1–1 and onto.

2. One can construct the Klein bottle KB using two smooth charts (U_i, h_i) for $i = 1, 2$ where $U_1 = U_2 = \mathbf{R}^2 - \{0\}$ such that the overlapping images are given by $V_{21} = V_{12} = \{ z \mid |z| < \frac{1}{2}$ or $|z| > 2 \}$ and the transition diffeomorphisms ψ_{ij} are both given by $\psi_{ij}(x, y) = (x, y)$ if $\sqrt{x^2 + y^2} > 2$ and $\psi_{ij}(x, y) = (x, -y)$ if $\sqrt{x^2 + y^2} < \frac{1}{2}$ What are the domains of the charts for the corresponding smooth atlas of the tangent space $T(KB)$, and what is the corresponding transition map for these charts?

3. A smooth curve γ from an open interval (a, b) to a smooth manifold M is said to be regular if its tangent vector at every point is nonzero, and a continuous curve γ from (a, b) to M is said to be *regularly piecewise smooth* if one can find a partition

$$a = s_0 < s_1 < \cdots < s_p = b$$

such that the restrictions of γ to the pieces $(s_0, s_1], [s_1, s_2] \ \cdots \ [s_{p-1}, s_p)$ all extend to regular smooth curves on open intervals containing the pieces.

(i) Prove that if M is connected, then every pair of points can be joined by a regular piecewise smooth curve $(-\varepsilon, 1 + \varepsilon)$; *i.e.*, for each $x, y \in M$ one can find such a curve γ so that $\gamma(0) = x$ and $\gamma(1) = y$.

(ii) Prove the following strengthening ot (i): Every pair of points can be joined by a regular smooth curve. [*Hint:* start with the conclusion of (i) and use the construction on the last three pages of `nicecurves.pdf` to smooth out the corner points.]

4. Assume we are in the setting of Additional Exercise I.4, where we are given two smooth n-manifolds M and N such that there is a diffeomorphism Φ from an open subset $U \subset M$ to an open subset $V \subset N$. Then we can form the quotient space $P = M \cup_{\Phi} N$, which is given by $M \amalg N$ modulo the equivalence relation generated by $x \equiv \Phi(x)$ for all $x \in U$. If this space P is Hausdorff, then the cited exercise yields a smooth structure on P which contains open subsets diffeomorphic to M and N.

Prove that the space $T(M) \cup_{T(\Phi)} T(N)$ is Hausdorff, and with the smooth structure on it given in the earlier exercise it is diffeomorphic to $T(P)$.

5. Let M be a smooth manifold, let $p : E \to M$ be a Hausdorff covering space projection, and take the smooth structure on E given in the lectures. Prove that $T(p) : T(E) \to T(M)$ is also a smooth covering space projection of the same type, and if $h : M \to M$ is a covering space (deck) transformation then so is $T(h)$.

4 : Mersions and embeddings

CONVENTION. The word *mersion* refers to a map which is either an immersion or a submersion. There is also a related concept of k-mersion, which is a smooth mapping $f : M \to N$ such that for each $x \in M$ the tangent space mapping $T(f)_x : T_x(M) \to T_{f(x)}(N)$ has constant rank k.

Lee, 4.10, 4.38, $4-6$, $4-8$, $4-12$, $4-13$

Additional exercises

1. (i) Prove that the composite of two smooth immersions is a smooth immerson.

(ii) Prove that the composite of two smooth submersions is a smooth submerson.

(iii) Prove that the composite of two smooth embeddings is a smooth embedding.

2. Let $f : M \to M'$ and $g : N \to N'$ be smooth mappings.

(i) Prove that if f and g are immersions then so is $f \times g : M \times N \to M' \times N'$.

(ii) Prove that if f and g are submersions then so is $f \times g : M \times N \to M' \times N'$.

(iii) Prove that if f and g are smooth embeddings then so is $f \times g : M \times N \to M' \times N'$.

3. Let X be the y-axis in the Cartesian plane, and let Y be the graph of $\sin \frac{1}{x}$ for $x > 0$. Prove that the map $X \amalg Y \to \mathbb{R}^2$ is an immersion but not an embedding; also show that the restrictions to the two pieces are embeddings.

4. Prove that there is no immersion from a compact n-manifold into $\mathbb{R}^n$.

5. Prove that there is no submersion from a compact n-manifold into $\mathbb{R}$. [*Hint:* Such a map attains a maximum value. What does this mean if we look at a smooth chart at a point where the maximum is attained?]

Note. If M is connected and noncompact, then one can always construct a smooth submersion $M \to \mathbb{R}$.

6. Given an immersion from a 1-connected compact smooth manifold to a smooth manifold of the same dimension, prove that it is a covering space projection. Does the statement remain true if the manifolds are not necessarily compact? Prove this or give a counterexample.

7. Let U and V be open in $\mathbb{R}^n$, let $f : U \to V$ be a smooth *surjective* immersion/submersion, and suppose that $g : V \to \mathbb{R}^q$ is a continuous map such that $g \circ f$ is smooth. Prove that g is also smooth.

8. A continuous map $f : A \to X$ is a *retract* if there is a continuous map $g : X \to A$ such that $g \circ f = \mathrm{id}_A$. Suppose that A and X smooth manifolds and f and g are smooth. Prove that f is an immersion.

9. Suppose that M is a noncompact smooth manifold and there is a smooth 1–1 immersion $f : M \to \mathbb{R}^N$. Prove that there is a smooth embedding $g : M \to \mathbb{R}^{N+1}$ such that $g[M]$ is a closed subset. [*Hint:* Recall that there is a proper map from M to $\mathbb{R}$.]

10. (i) Suppose that M and N are smooth manifolds. A smooth map $f : M \to N$ is said to be a *retract* if there is a smooth map $g : N \to M$ such that $g \circ f = \mathrm{id}_M$. Prove that a smooth retract is a smooth immersion.

(ii) A smooth map of smooth manifolds $r : N \to M$ is said to be a smooth *retraction* if there is a smooth map $j : M \to N$ such that $r \circ j = \mathrm{id}_M$. Prove that if r is a retraction, then the restriction of r to some neighborhood of $j(M)$ is a submersion.

(iii) A continuous map of topological spaces is said to be a continuous retract if it satisfies the condition in (i). Prove that if A and X are Hausdorff then j is a closed mapping. Why does this imply that j maps A homeomorphically onto its image? [*Hint:* To see that A is closed, show that it is the set of all points such that $x = j \circ r(x)$.]

11. Let $z : M \to T(M)$ be the map which sends each point $x \in M$ to the zero vector in the tangent space $T_x(M)$. Prove that z is a smooth embedding. [*Hint:* What does z look like in local coordinates, and why is $\tau_m \circ z$ the identity?]

12. Let $z : M \to T(M)$ be given as in the previous exercise. Prove that $z[M]$ is a strong deformation retract of $T(M)$ and τ_M is an associated deformation retraction.

13. Let $n_1, \cdots, n_k$ be positive integers and let N be their sum. Prove that there is a smooth embedding of $\prod_j S^{n_j}$ into S^{N+1}. [*Hint:* One always has smooth embeddings of $S^p \times \mathbb{R}^q$ in $\mathbb{R}^{p+q}$ and embeddings of $S^{q-1} \times \mathbb{R}$ in $\mathbb{R}^q$. Use these as part of an inductive argument.]

14. Suppose we have smooth maps $i : M \to N$ and $j : N \to L$ such that $j \circ i$ is a smooth embedding. Prove that i is a smooth embedding.

5 : Smooth submanifolds

Lee, 5 – 1, 5 – 3, 5 – 6, 5 – 7

Additional exercises

1.