EXERCISES FOR MATHEMATICS 205C

SPRING 2016 — Part 3

The headings denote chapters of the text for the course:

J. Lee, *Introduction to Smooth Manifolds* (Second Edition), Springer-Verlag, 2012.

Exercises which appear throughout the text are numbered in the form $m.n$, and exercises at the end of the chapters are numbered in the form $m - n$. Except when explicitly noted otherwise, it will suffice to prove exercises for manifolds without boundary.

7 . Lie groups

Lee, $7 - 1$ — $7 - 5$, $7 - 14$, $7 - 17$, $7 - 20$, $7 - 21$, $7 - 24$

Additional exercises

1. We have defined the affine group $\mathrm{Aff}(\mathbb{R}^n)$ to be the semidirect product of $GL(n, \mathbb{R})$ and $\mathbb{R}^n$ (as an additive abelian Lie group) with twisting homomorphism $GL(n, \mathbb{R}) \to \mathrm{Aut}(\mathbb{R}^n)$ given by the action of $GL(n, \mathbb{R})$ on $\mathbb{R}^n$ by invertible linear transformations.

(a) Prove that the affine group is isomorphic to the subgroup of $GL(n + 1, \mathbb{R})$ consisting of all matrices having the form

$$\begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix}$$

where $A \in GL(n, \mathbb{R})$ and $b \in \mathbb{R}^n$ (hence the bottom blocks have only one row). Also, prove that this subgroup is a closed Lie subgroup of $GL(n + 1, \mathbb{R})$. [*Hint:* Show that the matrix group in question is the zero set of some smooth function f into $\mathbb{R}^{n+1}$ and that $Df(I_{n+1})$ has maximum rank, where I_k is the identity $k \times k$ matrix.]

(b) Prove that the Galilean group is the closed Lie subgroup of all $(n + 1) \times (n + 1)$ matrices as above such that A is an orthogonal matrix.

2. Let $n \geq 2$ be an integer. The *conformal linear group* Conf_n is the set of all matrices $A \in GL(n, \mathbb{R})$ such that A preserves the angles between nonzero vectors. Equivalently, this group is defined by the identity

$$\frac{\langle Ax,\, Ay\rangle}{|Ax|\,|Ay|} \quad = \quad \frac{\langle x,\, y\rangle}{|x|\,|y|}$$

for each pair of nonzero vectors $x, y \in \mathbb{R}^n$.

(a) Prove that $A \in \mathrm{Conf}_n$ if and only if $A = rB$ where $r > 0$. [*Hint:* The ($\Leftarrow$) direction is very straightforward. To prove the ($\Rightarrow$) implication, consider the Gram matrix ${}^{\mathbf{T}}\!A\,A$, whose entries are the inner products of the columns of A. Why is this product a diagonal matrix with positive entries down the diagonal? Using the perpendicularity of the vectors $\mathbf{e}_i + \mathbf{e}_j$ and $\mathbf{e}_i - \mathbf{e}_j$ (for all $i \neq j$) and the fact that A is conformal, conclude that all the diagonal entries of ${}^{\mathbf{T}}\!A\,A$ are equal to some positive number c. Why is $B = c^{-1/2}\,A$ an orthogonal matrix?]

(b) Prove that the map $h : \mathbb{R} \times \mathrm{O}_n \to GL(n, \mathbb{R})$ sending (t, A) to $e^t A$ is a proper (hence closed) smooth embedding whose image is equal to Conf_n. In particular, this implies that the group Conf_n has a natural Lie group structure.

3. The purpose of this exercise is to determine the numbers of connected components for each of the groups $GL(n, \mathbb{R})$, $GL(n, \mathbb{C})$, $O(n)$ [the group of orthogonal $n \times n$ matrices, and $U(n)$ [the group of unitary $n \times n$ matrices.

(a) Let A and P be matrices in any one of these groups. Prove that A lies in the arc component of the identity if and only if PAP^{-1} does. [*Hint:* Suppose that $A(t)$ is a curve joining A to I, and consider $PA(t)P^{-1}$.

(b) Prove that $A \in GL(n, \mathbb{R})$ lies in the (arc) component of the identity if and only if its determinant is positive. [*Hints:* Since det is an onto continuous function into $\mathbb{R} - \{0\}$ and the latter has two components, a matrix with negative determinant cannot lie in the component of the identity. Suppose henceforth that the determinant of A is positive. Express A as a product of elementary matrices $P_1 \cdots P_k$ (for convenience, we expand the definition so that I is also an elementary matrix). If we are given a product B of elementary matrices $Q_1 \cdots Q_k$ such that, for each i, the matrices P_i and Q_i lie in the same arc component of $GL(n, \mathbb{R})$, explain why A and B lie in the same arc component of $GL(n, \mathbb{R})$. Now consider the various types of elementary matrices obtained from the identity by the operations (0) do nothing, (1) multiply a row by a positive constant, (2) multiply a row by -1, (3) add a multiple of one row to another, (4) interchange two rows. Why do elementary matrices of types $(0), (1), (3)$ in the arc component of the identity matrix. Next, consider the following 2×2 matrix identity:

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The left hand side represents a counterclockwise rotation through 90° and hence lies in the arc component of the identity; in fact, this is true if we replace $GL(n, \mathbb{R})$ by $O(n)$. Why does this imply that the two factors on the right lie in the same component of $GL(n, \mathbb{R})$? Use the preceding observations to show that every invertible matrix lies in the same arc component as a product of elementary matrices of types (0) and (2)? Recall that if I_2 is the 2×2 identity matrix, then $-I_2$ is rotation through 180° and hence lies in the identity component of $O(2)$. Finally, if $\det A > 0$ then the preceding shows that A lies in the same arc component as a product of an even number of elementary matrices having type (2). Why does this imply that A lies in the same arc component as the identity?]

(c) An orthogonal matrix has determinant ± 1. Prove that an orthogonal matrix A lies in the identity component of $O(n)$ if and only if $\det A = 1$. [*Hints:* By Appendix D in the document

`http://math.ucr.edu/~res/math205A-2014/gentop-notes.pdf`

every orthogonal matrix A is expressible as PBP^{-1} where P is an orthogonal matrix and B is a block sum of of 1×1 and 2×2 orthogonal matrices, so by (a) it suffices to prove the result for such block sums. One can check directly from the description of orthogonal 2×2 matrices in the preceding document that $O(2)$ has precisely two components and the identity component is the subgroup with determinant 1; note that the latter consists of the rotation matrices. Now note that if B is a block sum of the small matrices B_i and C_i is a comparable set of small orthogonal matrices with the same sizes, then the block sum C lies in the same arc component as B if we have $\det B_i = \det C_i$ for all i. Why does this imply that B lies in the arc component of an orthogonal matrix which is diagonal with entries ± 1? Finally, if $\det A = 1$ then there must be an even number of negative diagonal terms. Why does this imply that A lies in the same arc component as the identity?]

(d) Prove that $GL(n, \mathbb{C})$ is connected. [*Hints:* Why does it suffice to prove this for matrices in Jordan form? Recall that the multiplicative group $\mathbb{C} - \{0\}$ is connected and the diagonal entries of an invertible matrix in Jordan form are all nonzero.]

(d) Prove that $U(n)$ is connected. [*Hints:* By the Spectral Theorem, if A is a unitary matrix then there is another unitary matrix P such that A is expressible as PBP^{-1} where B is a diagonal matrix whose nonzero entries all lie in S^1.]

4. The following exercise uses the definition and elementary properties of the matrix exponential map exp, which is defined by the usual power series for the ordinary exponential function e^x. Background information can be found in `expmatrix.pdf`. — Also, if G is a group and $g \in G$, then χ_g will denote the inner automorphism $x \longrightarrow gxg^{-1}$. Note that this automorphism is a diffeomorphism if G is a Lie group.

If G is a Lie group and $\mathcal{L}(G)$ denotes its tangent space, then the *adjoint representation* is a homomorphism

$$\mathrm{ad}_G : G \longrightarrow GL\,(\,\mathcal{L}(G))$$

given by $\mathrm{ad}_G(A) \;=\; T(\chi_g)(A)$ for $A \in \mathcal{L}(G)$. Suppose that $A(t) = \exp(tA)$ and $B \in \mathcal{L}(G)$, and consider the smooth curve

$$\gamma(t) \quad = \quad \chi_{\exp(tA)}(B) \quad = \quad \exp(tA)\,B\,\exp(-tA)\ .$$

Prove that $\gamma'(0)$ is the commutator $[A, B]$. — What this means is that the lack of commutativity of the general linear group is somehow measured by a lack of commutativity in the Lie algebra of that group. [*Hint:* Use the power series expansion of $\exp(tA)$ and its inverse, and notice that higher order terms will not affect the answer.]

5. This exercise also uses the matrix exponential map.

(a) Suppose that $H \subset G$ is an embedded connected Lie subgroup of G and we have the associated inclusion of tangent spaces at the identity from $\mathcal{L}(H)$ to $\mathcal{L}(G)$. Prove that $\mathcal{L}(H)$ is a Lie subalgebra of $\mathcal{L}(G)$.

(b) Suppose in addition that H is a normal subgroup. In this case prove that $\mathcal{L}(H)$ is a Lie ideal of $\mathcal{L}(G)$; *i.e.*, If $B \in \mathcal{L}(H)$ and $A \in \mathcal{L}(G)$, then $[A, B] \in \mathcal{L}(H)$.

NOTE. Both of these inclusions hold more generally, and in fact there are important partial converses (these are special cases of Lie's results).

6. Let $SU(n) \subset U(n)$ denote the set of matrices whose determinant is 1.

(a) Prove that $SU(n) \subset U_n$ is a closed Lie subgroup. [*Hint:* Imitate the proof for $SL(n, \mathbb{R}) \subset GL(n, \mathbb{R})$.]

(b) Prove that $U(n) \cong S^1 \times SU(n)$ as smooth manifolds but not as Lie groups. [*Hint:* View S^1 as the diagonal matrices with variable entries in the upper left hand corner. To show that the map is not a group isomorphism, you may use the following fact: The center of $SU(n)$ is finite. There is a proof of this result in `centers.pdf`.]

7. Prove that the 2×2 diagonal matrix

$$C \quad = \quad \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix}$$

does not lie in the image of the exponential map. [*Hint:* Suppose it did, write $C = \exp(A)$, and consider $B = \exp\left(\frac{1}{2}A\right)$. Why do we then have $B^2 = C$ and $CB = BC$? Derive a contradiction by showing that there is no matrix with these properties.]

8. Let $\mathcal{H}(3)$ be the *Heisenberg group*, which consists of the upper triangular 3×3 matrices with ones down the diagonal. Show that $\mathcal{H}(3)$ is a nonabelian Lie group which is diffeomorphic to $\mathbb{R}^3$ (where the group operation is matrix multiplication). More generally, if $\mathcal{H}(n)$ is the set of upper triangular $n \times n$ matrices whose diagonal entries are ones and whose remaining entries are zero except in the first row and last column, prove that $\mathcal{H}(n)$ is a nonabelian Lie group with respect to matrix multiplication and it is diffeomorphic to $\mathbb{R}^{2n-3}$ as a smooth manifold.

NOTE. Finding examples of **compact** Lie groups which are diffeomorphic, but not isomorphic as Lie groups, is considerably more difficult. Examples of this sort are given in Theorem 9.4 from the following paper:

> **P. Baum and W. Browder,** *The cohomology of quotients of classical groups*, Topology **3** (1965), 305–336.

9. Let $\mathcal{L}$ be a Lie algebra over the real numbers with multiplication denoted by $[\cdots,\cdots]$, and let $a \in \mathcal{L}$. Prove that the map $D_a : \mathcal{L} \to \mathcal{L}$ defined by $D_a(x) = [a, x]$ is a derivation of $\mathcal{L}$. — These derivations are called *inner derivations* of the Lie algebra $\mathcal{L}$. [*Hint:* Use the Jacobi Identity.]

8 . Vector fields

Lee, $8-1$ — $8-3$, $8-11$, $8-16$, $8-19$

Additional exercises

1. Suppose that X and Y are smooth vector fields on an open set in some Euclidean space $\mathbb{R}^b$, and let D_X and D_Y be the corresponding derivations on the ring of smooth functions $\mathbf{C}^\infty(M)$. Give an example to show that the composite $D_X D_Y$ is not necessarily a derivation.

2. Let X and Y be the vector fields in the plane defined by the vector-valued smooth functions (x, xy) and (y^2, xy) respectively. Compute the Lie bracket $[X, Y]$.

3. Find the Lie brackets of the following pairs of vector fields on $\mathbb{R}^2$ (we write ∂_u for $\frac{\partial}{\partial u}$ to save space):

(*i*) $x\,\partial_y + y\,\partial_x$ and $x\,\partial_x + y\,\partial_y$.

(*ii*) $-y\,\partial_x + x\,\partial_y$ and $y\,\partial_x + x\,\partial_y$.

4. In the notation of the preceding exercise, show that if $g(x, y)$ is a smooth function on $\mathbb{R}^2$ such that $[g(x, y)\partial_x, \partial_y]$ is the zero vector field (*i.e.*, the vector fields *commute*), then $g(x, y)$ is a function of x alone. [*Hint:* What conclusion can be drawn if the partial derivative with respect to one variable is identically zero?]

5. Find the Lie brackets of the following pairs of vector fields on $\mathbb{R}^3$ (as before, let $\partial_u = \frac{\partial}{\partial u}$):

(*i*) $y\,\partial_z - 2xy^2\,\partial_y$ and ∂_y.

(*ii*) $x\,\partial_x + y\,\partial_y$ and $x\,\partial_y + y\,\partial_z$.

6. Suppose that a smooth function f satisfies $[fX, Y] = f[X, Y]$ for all vector fields X and Y. What can one say about f?

7. Let X be a vector field on M which is nowhere zero. Prove that the set of all products $g \cdot X$ (where g is a smooth function) is a Lie subalgebra of the space of all vector fields; in other words, it is a $\mathbf{C}^\infty(M)$-submodule and it is closed under taking Lie brackets.

8. Let φ be a diffeomorphism from one smooth manifold M to another smooth manifold N, and let $\mathbf{VF}(P)$ denote the vector space of smooth vector fields on a smooth manifold P. Then φ defines a map φ_* from $\mathbf{VF}(M)$ to $\mathbf{VF}(N)$ as follows:

$$\varphi_*(X) \quad = \quad T(\varphi) \circ X \circ \varphi^{-1}$$

In the lectures it was shown that φ_* defines an isomorphism of Lie algebras. Verify the following additional properties of this construction:

(a) If φ is an identity mapping, then so is φ_*.

(b) We have $(\varphi_*)^{-1} \;=\; \left(\varphi^{-1}\right)_*$.

(c) If ψ is a diffeomorphism from N to Q, then we have $(\psi \circ \varphi)_* \;=\; \psi_* \circ \varphi_*$.

9. Suppose that $X : M \to T(M)$ is a smooth vector field. Explain why $T(X) : T(M) \to T(T(M))$ is usually not a smooth vector field. It will suffice to consider the case where M is an open subset U of some $\mathbb{R}^n$. [*Hint:* Consider the standard diffeomorphisms

$$T(T(U)) \;\cong\; T(U \times \mathbb{R}^n) \;\cong\; (U \times \mathbb{R}^n) \times (\mathbb{R}^n \times \mathbb{R}^n) \qquad \text{and}$$

$$T(U \times \mathbb{R}^n) \;\cong\; T(U) \times T(\mathbb{R}^n) \;\cong\; U \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$$

and explain why the two identifications of $T(U \times \mathbb{R}^n)$ with $U \times (\mathbb{R}^n)^3$ differ by a permutation of coordinates. Think about what would happen if U were open in $\mathbb{R}^m$ where $m \neq n$.]

10. Suppose that U is open in $\mathbb{R}^n$ and $f : U \to \mathbb{R}$ is smooth. The **gradient** of f, written as usual by ∇f, is the vector field $x \to (x, \mathbf{F}(x))$ where $\mathbf{F} : U \to \mathbb{R}^n$ is the smooth function whose coordinates are the partial derivatives of f with respect to the appropriate variables (just as in multivariable calculus).

(a) Suppose that 0 is a regular value of f and $H \subset \mathbb{R}^n$ is the inverse image of $\{0\}$. Show that the $\nabla f(x)$ is perpendicular to $T_x(H) \subset \{x\} \times \mathbb{R}^n$ for all $x \in H$.

(b) In the setting above, suppose that f is never zero and X is a vector field. Prove that X is perpendicular to ∇f if and only if $Xf = 0$.

(c) Still in the setting above, suppose that X and Y are vector fields and both are perpendicular to ∇f. Prove that $[X, Y]$ is also perpendicular to ∇f.