EXERCISES FOR MATHEMATICS 205C

SPRING 2016 — Part 4

More additional exercises for Chapter 5

These exercises deal with the material in `newnotes05c.pdf`.

1. If U is open in $\mathbb{R}^n$, then a riemannian metric on U is given by a map of the form

$$\gamma : U \times \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R} , \qquad \gamma(x; A, B) = {}^{\mathbf{T}}B\, G(x)\, A$$

where $\mathbb{R}^n$ is viewed as the space of $n \times 1$ column vectors and $G(x)$ is a smooth function of x whose image lies in the subspace of $n \times n$ matrices which are symmetric and positive definite. Using the Principal Minors test for positive definite matrices, prove the following: Given G, there is some $\delta > 0$ such that if $H(x)$ is a symmetric matrix valued function satisfying $|h_{i,j}(x) - g_{i,j}(x)| < \delta$ for all x, i and j, then the map θ sending $(x; A, B)$ to ${}^{\mathbf{T}}B\, H(x)\, A$ also defines a riemannian metric on U.

2. Suppose that $g : T(M) \times_M T(M) \to \mathbb{R}$ is a riemannian metric on M and $\varphi : M \to N$ is a diffeomorphism.

(a) Explain why the map $T_2(\varphi) : T(M) \times_M T(M) \to T(N) \times_N T(N)$ which sends $(A, B) \in T_p(M) \times T_p(M)$ to

$$(\, T(\varphi)A,\ T(\varphi)B\,) \in T_{\varphi(p)}(N) \times T_{\varphi(p)}(N)$$

(for each $p \in M$) is a diffeomorphism. [*Hint:* As usual, look locally.]

(b) In the setting above, show that the composite $\varphi_*(g) = g \circ T_2(\varphi)^{-1}$ defines a riemannian metric on N such that for each $p \in M$ the map sends $T_p(M)$ to $T_{\varphi(p)}(N)$ by an isometry of inner product spaces. Furthermore, prove that if g_1, g_2 and $g_1 + g_2$ are riemannian metrics then $\varphi_*(g_1 + g_2) = \varphi_*(g_1) + \varphi_*(g_2)$, and if $s > 0$ is a constant, then $\varphi_*(s \cdot g_1) = s \cdot \varphi_*(g_1)$.

(c) Prove that if φ is the identity map then so is φ_*, and if $\psi : N \to P$ is also a diffeomorphism then $(\psi \circ \varphi)_* = \psi_* \circ \varphi_*$. Why do these imply that φ_* induces an isomorphism from the space of riemannian metrics on M to the space of riemannian metrics on N?