Stereographic projection and inverse geometry

The conformal property of stereographic projections can be established fairly efficiently using the concepts and methods of inverse geometry. This topic is relatively elementary, and it has important connections to complex variables and hyperbolic (= Bolyai-Lobachevsky noneuclidean) geometry.

Definition. Let $r > 0$ be a real number, let $y \in \mathbb{R}^n$, and let $S(r; y)$ be the set of all points $x \in \mathbb{R}^n$ such that $|x - y| = r$. The *inversion map with respect to the sphere* $S(r; y)$ is the map T on $\mathbb{R}^n - \{0\}$ defined by the formula

$$T(x) = y \ + \ \frac{r^2}{|x-y|} \cdot \big(x - y\big) \ .$$

Altternatively, $T(x)$ is defined so that $T(x) - y$ is the unique positive scalar multiple of $x - y$ such that

$$\big|T(x) - y\big) \cdot |x - y\big) = r^2 \ .$$

Another way of saying this is that inversion interchanges the exterior points to $S(r; y)$ and the interior points with the center deleted. If $r = 1$ and $y = 0$ then inversion simply takes a nonzero vector x and sends it to the nonzero vector pointing in the same direction with length equal to the reciprocal of $|x|$ (this should explain the term "inversion").

If $n = 2$ then inversion corresponds to the *conjugate* of a complex analytic function. Specifically, if a is the center of the circle and r is the radius, then inversion is given in complex numbers by the formula

$$T(z) \quad = \quad r^2 \cdot \big(\overline{z} - a\big)^{-1} \quad = \quad r^2 \cdot \overline{\big(z - \overline{a}\big)^{-1}}$$

where the last equation holds by the basic properties of complex conjugation. If $r = 1$ and $a = 0$ then inversion is just the conjugate of the analytic map sending z to z^{-1}.

The geometric properties of the analytic inverse map on the complex plane are frequently discussed in complex variables textbooks. In particular, this map has nonzero derivative wherever the function is defined, and accordingly the map is conformal. Furthermore, the map $z \to z^{-1}$ is an involution (its composite with itself is the identity), and it sends circles not containing 0 to circles of the same type. In addition, it sends lines not containing the origin into circles containing the origin and vice versa (*Note:* This means that 0 lies on the circle itself and NOT that 0 is the center of the circle!). Since complex conjugation sends lines and circles to lines and circles, preserves the angles at which curves intersect and also sends 0 to itself, it follows that the inversion map with respect to the unit circle centered at 0 also has all these properties).

It turns out that all inversion maps have similar properties. In particular, they send the every point of sphere $S(r; y)$ to itself and interchange the exterior points of that sphere with all of the interior points except y (where the inversion map is not defined), they preserve the angles at which curves intersect, they are involutions, they send hyperspheres not containing the central point y to circles of the same type, and they send hyperplanes not containing $\{y\}$ into hyperspheres containing $\{y\}$ and vice versa. We shall limit our proofs to the properties that we need to study stereographic projections; the reader is encouraged to work out the proofs of the other assertions.

Relating stereographic projections and inversions

We begin by recalling(?) some simple observations involving isometries and similarity transformations from $\mathbb{R}^n$ to itself.

FACT 1. *If $b \in \mathbb{R}^n$ and F is translation by b (formally, $F(x) = x + b$), then F is an isometry from $\mathbb{R}^n$ to itself and F sends the straight line curve from x to y defined by*

$$\alpha(t) = ty + (1-t)x$$

to the straight line curve from $F(x)$ to $F(y)$ defined by

$$\alpha(t) = tF(y) + (1-t)F(x) \ .$$

Definition. If $r > 0$ then a *similarity transformation with ratio of similitude* r on a metric space X is a 1–1 correspondence f from X to itself such that $\mathbf{d}\big(f(x), f(y)\big) = r \cdot \mathbf{d}(x, y)$ for all $x,\ y \in X$.

Note that every isometry (including every identity map) is a similarity transformation with ration of similitude 1 and conversely, the inverse of a similarity transformation with ratio of similitude r is a similarity transformation with ratio of similitude r^{-1}, and the composite of two similarity transformations with ratios of similitude r and s is a similarity transformation with ratio of similitude rs. Of course, if $r > 0$ then the invertible linear transformation rI on $\mathbb{R}^n$ is a similarity transformation with ratio of similitude r.

FACT 2. *In $\mathbb{R}^n$ every similarity transformation with ratio of similitude r satisfying $F(0) = 0$ has the form $F(x) = rA(x)$ where A is given by an $n \times n$ orthogonal matrix.*

FACT 3. *In $\mathbb{R}^n$ every similarity transformation F is conformal; specifically, if α and β are differentiable curves in $\mathbb{R}^n$ that are defined on a neighborhood of $0 \in \mathbb{R}$ such that $\alpha(0) = \beta(0)$ such that both $\alpha'(0)$ and $\beta'(0)$ are nonzero, then the angle between $\alpha'(0)$ and $\beta'(0)$ is equal to the angle between $[F \circ \alpha]'(0)$ and $[F \circ \beta]'(0)$.*

The verification of the third property uses Facts 1 and 2 together with the additional observation that if A is given by an orthogonal transformation then A preserves inner products and hence the cosines of angles between vectors.

The key to relating inversions and stereographic projections is the following result:

PROPOSITION. *Let $e \in \mathbb{R}^n$ be a unit vector, and let T be inversion with respect to the sphere $S(1;0)$. Then T interchanges the hyperplane defined by the equation $\langle x, e\rangle = -1$ with the nonzero points of the sphere $S(\frac{1}{2};\ -\frac{1}{2}e)$.*

Proof. By definition we have

$$T(x) = \frac{1}{\langle x, x\rangle} \cdot x$$

and therefore the proof amounts to finding all x such that

$$\left| T(x) + \tfrac{1}{2}e \right| = \tfrac{1}{2}.$$

This equation is equivalent to

$$\tfrac{1}{4} = \left| T(x) + \tfrac{1}{2}e \right|^2 = \langle T(x) + \tfrac{1}{2}e,\ T(x) + \tfrac{1}{2}e\rangle$$

and the last expression may be rewritten in the form

$$\langle T(x), T(x)\rangle + \langle T(x), e\rangle + \frac{1}{4} =$$

$$\frac{\langle x, x\rangle}{\langle x, x\rangle^2} + \frac{\langle x, e\rangle}{\langle x, x\rangle} + \frac{1}{4}$$

which simplifies to

$$\frac{1}{\langle x, x\rangle} + \frac{\langle x, e\rangle}{\langle x, x\rangle} + \frac{1}{4}\ .$$

Our objective was to determine when this expression is equal to $\frac{1}{4}$, and it follows immediately that the latter is true if and only if $1 + \langle x, e\rangle = 0$; *i.e.*, it holds if and only if $\langle x, e\rangle = -1$.

COROLLARY. *Let e be as above, and let W be the $(n-1)$-dimensional subspace of vectors that are perpendicular to e. Then the stereographic projection map from $S(1;0) - \{e\}$ to W is given by the restriction of the composite*

$$G \circ t \circ H$$

to $S(1;0) - \{e\}$, where H is the similarity transformation $H(u) = \frac{1}{2}(u - e)$ and G is the translation isometry $G(v) = v + e$.

Proof. It will be convenient to talk about the *closed ray* starting at a vector a and pasing through a vector b; this is the image of the parametrized curve

$$\gamma(t) = (1-t)a + tb = a + t(b-a)$$

where $t \geq 0$.

First note that the composite $T \circ H$ sends the ray starting at the point e and passing throug a point x with $\langle x, e\rangle < 1$ to the ray starting at e and passing through the point $H(x)$, and the latter satisfies $\langle H(x), e\rangle < 0$. This is true for the mapping H by Fact 1 and the equation $H(e) = 0$, and it is true for $T \circ H$ because T is inversion with respect to a sphere centered at 0. Furthermore, by construction H sends the sphere $S(1; 0)$ to $S\frac{1}{2};\ -\frac{1}{2}e)$, and it also sends the hyperplane P defined by $\langle x, e\rangle = -1$ to itself.

By construction, stereographic construction sends the point $y \in S(1; 0) - \{e\}$ to the point $z \in W$ such that $z - e$ is the unique point at which the ray starting at e and passing through y meets the hyperplane P, and inversion sends the point $\eta \in S(\frac{1}{2};\ -\frac{1}{2}e) - \{0\}$ to the unique point α at which the ray starting at 0 and passing through η meets the hyperplane P.

Combining these, we see that $T \circ H$ maps y to the unique point where the ray passing through 0 and y meets the hyperplane P. This point may be written uniquely in the form $w - e$ where $w \in W$, and in fact we have $w = G \circ T \circ H(\mathrm{y})$. On the other hand, by the preceding paragraph we also know that w is given by the stereographic projection.

The conformal property for inversions

The following is an immediate consequence of the Chain Rule and Fact 3 stated abbove:

PROPOSITION. *Let U be open in $\mathbb{R}^n$, let $x \in U$ and let $f : U \to \mathbb{R}^n$ be a $\mathbf{C}^1$ mapping. Then f is conformal at x if $Df(x)$ is a nonzero scalar multiple of an orthogonal transformation.*

The key observation behind the proposition is that if $\gamma : (-\delta, \delta) \to U$ is a differentiable curve with $\gamma(0) = x$ then

$$[f \circ \gamma]'(0) = Df(x)[\gamma'(0)]$$

by the Chain Rule.

We are now ready to prove the result that we wanted to establish.

THEOREM. *Every inversion map is conformal.*

Proof. It is convenient to reduce everything to the case where the sphere is $S(1; 0)$. Given an arbitrary sphere $S(r; y)$ there is a similarity transformation sending $S(r; y)$ to $S(1; 0)$ that is defined by the formula $F(x) = r^{-1}(x - y)$, and if T' and T are the associated inversions then we have

$$T' = F^{-1} \circ T \circ F\ .$$

By construction $DF(x) = r^{-1}I$ for all x and therefore it follows that $DT' = DT$ (as usual, "D" denotes the derivative of a function). Therefore, by the proposition it will suffice to show that DT is always a scalar multiple of an orthogonal map.

Let $x \neq 0$ and write an arbitrary vector $v \in \mathbb{R}^n$ as a sum $v = cx + u$ where $c \in \mathbb{R}$ and $\langle x, u\rangle = 0$. We have already noted that $T(x) = \rho(x)^{-2} \cdot x$ where $\rho(x) = |x|$, and we wish to use this in order to compute the value of $DT(x)$ at some vector $h \in \mathbb{R}^n$. The appropriate generalization of the Leibniz Rule for products and elementary multivariable calculus show that

$$DT(x)[h] = \rho(x)^{-2}h + \langle(\nabla[\rho(x)]^{-2}, h\rangle \cdot x$$

where

$$\nabla[\rho(x)]^{-2} = -[\rho(x)]^{-4} \cdot \nabla[\rho(x)]^{2} = -2[\rho(x)]^{-4}x \ .$$

If $\langle x, h\rangle = 0$ this shows that $DF(x)[h] = [\rho(x)]^{-2}h$, while if $h = x$ it follows that $DF(x)[x] = -[\rho(x)]^{-2}x$. In particular, this shows that there is an orthonormal basis for $\mathbb{R}^n$ consisting of eigenvectors with associated eigenvalues $\pm\rho(x)^{-2}$, and therefore it follows that $DT(x)$ is a positive scalar multiple of an orthogonal map, which in turn implies that T is conformal at x. Since x was arbitrary, this proves the theorem.