For compact manifolds, we have stronger results:

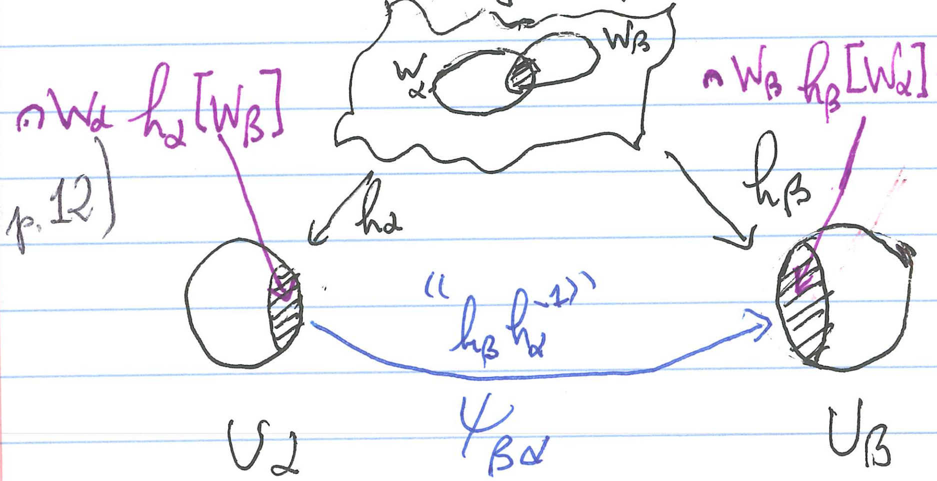
THEOREM. If M^n is a compact connected topological n-manifold, then $\pi_1(M^n, p)$ is a finitely presented group. Conversely, if $n \geq 4$ then every finitely presented group is $\pi_1(M^n, p)$ for some compact connected n-manifold M^n.

COORDINATE CHARTS. If M^n is a topological n-manifold, then a (topological) coordinate chart is a homeomorphism $h_\alpha : W_\alpha \to U_\alpha$ where W_α is open in M^n and U_α is open in $\mathbb{R}^n$.

Clearly M^n has an open covering $\{W_\alpha\}$ such that for each α we have a coordinate chart $h_\alpha : W_\alpha \to U_\alpha$.

Note that the change of coordinate maps are 1-1 onto continuous and open (hence are homeos.)

(Lee, p. 12)



The proper name for $\psi_{\beta\alpha}$ is

$$\left(h_\beta[W_\alpha \cap W_\beta] \,\middle|\, h_\beta \,\middle|\, W_\alpha \cap W_\beta\right) \circ \left(W_\alpha \cap W_\beta \,\middle|\, h_\alpha^{-1} \,\middle|\, h_\alpha[W_\alpha \cap W_\beta]\right)$$

A collection of charts $\{h_\alpha : W_\alpha \to U_\alpha\}$ is an <u>atlas</u> if $\{W_\alpha\}$ is an open covering of M^n.

<u>EXAMPLE.</u> $S^n \subseteq \mathbb{R}^{n+1}$ with stereographic projections $h_\pm : S^n - \{\pm e_{n+1}\} \longrightarrow \mathbb{R}^n$. As in Appendix B of gen-top. gentop-notes.pdf and in

<u>stereopic 1.pdf</u>
<u>stereopic 2.pdf</u>
<u>invstereo.pdf</u>

the transition map $\psi_{-+} : \mathbb{R}^n - \{0\} \longrightarrow \mathbb{R}^n - \{0\}$ is the classical inverse ~~map~~ geometry transformation

$$v \longmapsto \frac{4}{|v|^2} v \quad (\text{recall } v \neq 0)$$

so ψ_{-+} has coordinates with ∞ differentiable partial derivatives. Note that $(\psi_{-+})^{-1} = \psi_{-+}$.

One definition of a smooth structure

An atlas $\mathcal{A} = \{(h_\alpha : W_\alpha \to U_\alpha)\}$ is said to be a <u>smooth atlas</u> if the maps $\varphi_{\beta\alpha}$ are smooth and have smooth inverses (all coordinates have continuous partial derivatives of all orders). A smooth structure on a topological manifold M^n is a maximal atlas.

<u>Lee, Prop. 1.17</u> Every smooth atlas is contained in a maximal atlas, and it consists of all charts $h_\gamma : W_\gamma \to U_\gamma$ such that $\varphi_{\gamma\alpha}$ is smooth and has a smooth inverse.

<u>Idea of proof</u> Need to show that if $h_\gamma : W_\gamma \to U_\gamma$ and $h_\delta : W_\delta \to U_\delta$ are in the maximal atlas then $\varphi_{\delta\gamma}$ is smooth and has a smooth inverse. This only makes sense if $W_\gamma \cap W_\delta = \phi$, so let p lie in this intersection. The original atlas contains an α so that $p \in W_\alpha$. Informally we then have

$$\psi_{\delta\gamma} = h_\delta \circ h_\gamma^{-1} = \underbrace{h_\delta h_\alpha^{-1}}_{\text{Smooth}} \underbrace{h_\alpha h_\gamma^{-1}}_{\text{Smooth}} \text{ on}$$

$h_\gamma[W_\gamma \cap W_\delta \cap W_\alpha]$. Hence $\psi_{\delta\gamma}$ is smooth there.

Now a mapping is smooth $\Leftrightarrow$ its restriction to every set in an open covering is smooth. Since the sets $h_\gamma[W_\gamma \cap W_\delta \cap W_\alpha]$ cover $h_\gamma[W_\gamma \cap W_\delta]$, this means $\psi_{\delta\gamma}$ is smooth. To get smoothness of the inverse, note that $\psi_{\delta\gamma} = \psi_{\gamma\delta}^{-1}$, so the preceding also yields smoothness of the inverse. ∎

Open Subsets of Smooth Manifolds.

If $(M^n, \mathcal{O}_{max})$ is a smooth manifold and $U \subseteq M$ is open, then the set of all charts in $\mathcal{O}_{MAX}$ with domains contained in U will be a maximal smooth atlas for U.