CORRECTIONS TO
Introduction to Smooth Manifolds (Second Edition)

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(6/23/13) **Page 23, two lines below the first displayed equation:** Change "any subspace $S \subseteq V$" to "any k-dimensional subspace $S \subseteq V$."

(2/22/15) **Page 29, proof of Theorem 1.46, second paragraph, line 4:** Change $\varphi(U \cap V)$ to $\psi(U \cap V)$.

(10/8/15) **Page 30, Problem 1-6:** Interpret the formula for F_s to mean $F_s(0) = 0$ when $s \leq 1$.

(5/17/14) **Page 45, second paragraph:** Replace the last sentence of that paragraph with the following: "If N has empty boundary, we say that a map $F\colon A \to N$ is ***smooth on A*** if it has a smooth extension in a neighborhood of each point: that is, if for every $p \in A$ there exist an open subset $W \subseteq M$ containing p and a smooth map $F\colon W \to N$ whose restriction to $W \cap A$ agrees with F. When $\partial N \neq \varnothing$, we say $F\colon A \to N$ is smooth on A if for every $p \in A$ there exist an open subset $W \subseteq M$ containing p and a smooth chart (V, ψ) for N whose domain contains $F(p)$, such that $F(W \cap A) \subseteq V$ and $\psi \circ F|_{W \cap A}$ is smooth as a map into $\mathbb{R}^n$ in the sense defined above (i.e., it has a smooth extension in a neighborhood of each point)."

(7/23/14) **Page 45, last displayed equation:** The first $=$ sign should be $\subseteq$.

(11/17/12) **Page 56, first displayed equation:** Change $d\iota(v)_p$ to $d\iota_p(v)$.

(1/26/15) **Page 68, proof of Proposition 3.21:** Insert the following sentence at the beginning of the proof: "Let $n = \dim M$ and $m = \dim N$." Then in the second sentence, change (3.9) to (3.10). Finally, in the displayed equation, change F^n to F^m (twice).

(11/17/12) **Page 70, two lines above Corollary 3.25:** Change "Proposition 3.23" to "Proposition 3.24."

(3/5/15) **Page 76, Problem 3-8:** Add the following remark: "(For $p \in \partial M$, we need to allow curves with domain $[0, \varepsilon)$ or $(-\varepsilon, 0]$ and to interpret the derivatives as one-sided derivatives.)"

(5/4/13) **Page 96, Problem 4-3:** This problem probably needs a better hint. First, to get a good result, you'll have to add the assumption that $\ker dF_p \not\subseteq T_p \partial M$. After choosing smooth coordinates, you can assume $M \subseteq \mathbb{H}^m$ and $N \subseteq \mathbb{R}^n$, and extend F to a smooth function $\widetilde{F}$ on an open subset of $\mathbb{R}^m$. If $\operatorname{rank} F = r$, show that there is a coordinate projection $\pi : \mathbb{R}^n \to \mathbb{R}^r$ such that $\pi \circ \widetilde{F}$ is a submersion, and apply the rank theorem to $\pi \circ \widetilde{F}$ to find new coordinates in which $\widetilde{F}$ has a coordinate representation of the form $\widehat{F}(x, y) = (x, R(x, y))$. Then use the rank condition to show that $R|_M$ is independent of y.

(9/8/15) **Page 100, proof of Proposition 5.4, next-to-last line:** Change "It a homeomorphism" to "It is a homeomorphism."

(7/15/15) **Page 120, proof of Proposition 5.46:** At the beginning of the proof, insert this sentence: "Let $F\colon D \hookrightarrow M$ denote the inclusion map."

(4/17/13) **Page 132, proof of Lemma 6.13, second paragraph:** This argument does not apply when $\partial M \neq \varnothing$, because in that case $M \times M$ is not a smooth manifold with boundary. Instead, we can consider the restrictions of κ to $(M \times \operatorname{Int} M) \smallsetminus \Delta_M$ and to $(M \times \partial M) \smallsetminus \Delta_M$ (both of which are smooth manifolds with boundary), and note that there is a point $[v] \in \mathbb{RP}^{N-1}$ that is not in the image of τ or either of these restrictions of κ. [Thanks to David Iglesias Ponte for suggesting this correction.]

(7/3/15) **Page 134, displayed equations two-thirds of the way down the page:** In the definition of E_i, there's an "$i - i$" that should be "$i - 1$." It should read $E_i = f^{-1}\big([b_{i-1}, a_{i+1}]\big)$.

(11/25/12) **Page 145, statement of Corollary 6.33:** After "immersed submanifold," insert "with $\dim S = \dim M$."

(11/25/12) **Page 148, Problem 6-13:** Delete part (c). [This statement is simply wrong. It is true with the added hypothesis that F' is an embedding, but then it's essentially just a restatement of part (b).]

(8/26/14) **Page 169, first line:** Change $\widetilde{G}$ to G.

(9/17/14) **Page 173, Problem 7-21:** Replace the first sentence by "Prove that the groups in Problem 7-20 are isomorphic to direct products of the indicated groups in cases (a) and (c) if and only if n is odd, and in cases (b) and (d) if and only if $n = 1$."

(11/17/12) **Page 196, proof of Proposition 8.45, next-to-last line:** Should read "$F_* \circ \left(F^{-1}\right)_* = \left(F \circ F^{-1}\right)_* = \mathrm{Id}_{\mathrm{Lie}(H)}$ and $\left(F^{-1}\right)_* \circ F_* = \mathrm{Id}_{\mathrm{Lie}(G)}$."

(12/2/15) **Page 217, Fig. 9.7:** Both occurrences of φ should be Φ.

(12/2/15) **Page 219, second displayed equation:** Change "$V^j(0,p) = 0$" to "$\Phi^j(0,p) = 0$."

(12/2/15) **Page 219, two lines below (9.12):** Here and in the rest of the paragraph, change p_0 to p_1 (seven times) to avoid confusion with the prior unrelated use of p_0 in this proof.

(8/19/14) **Page 223, proof of Theorem 9.26:** There's a gap in this proof, because it is not necessarily the case that $M(a)$ is a regular domain in $\operatorname{Int} M$. To correct the problem, we have to choose our collar neighborhood more carefully. Replace the first sentence of the proof by the following paragraph:

"Theorem 9.25 shows that ∂M has a collar neighborhood C_0 in M, which is the image of a smooth embedding $E_0\colon [0,1) \times \partial M \to M$ satisfying $E_0(0,x) = x$ for all $x \in \partial M$. Let $f\colon M \to \mathbb{R}^+$ be a smooth positive exhaustion function. Note that $W = \{(t,x) : f(E_0(t,x)) > f(x) - 1\}$ is an open subset of $[0,1) \times \partial M$ containing $\{0\} \times \partial M$. Using a partition of unity as in the proof of Theorem 9.20, we may construct a smooth positive function $\delta\colon \partial M \to \mathbb{R}$ such that $(t,x) \in W$ whenever $0 \le t < \delta(x)$. Define $E\colon [0,1) \times \partial M \to M$ by $E(t,x) = E_0(t\delta(x), x)$. Then E is a diffeomorphism onto a collar neighborhood C of ∂M, and by construction $f(E(t,x)) > f(x) - 1$ for all $(t,x) \in [0,1) \times \partial M$. We will show that for each $a \in (0,1)$, the set $E([0,a] \times \partial M)$ is closed in M. Suppose p is a boundary point of $E([0,a] \times \partial M)$ in M; then there is a sequence $\{(t_i,x_i)\}$ in $[0,a] \times \partial M$ such that $E(t_i,x_i) \to p \in M$. Then $f(E(t_i,x_i))$ remains bounded, and thus $f(x_i) < f(E(t_i,x_i)) + 1$ also remains bounded. Since ∂M is closed in M, $f|_{\partial M}$ is also an exhaustion function, and therefore the sequence $\{x_i\}$ lies in some compact subset of ∂M. Passing to a subsequence, we may assume $(t_i,x_i) \to (t_0,x_0)$, and therefore $p = E(t_0,x_0) \in E([0,a] \times \partial M)$."

Then at the end of the first paragraph of the proof, add the following sentences:

"To see that $M(a)$ is a regular domain, note first that it is closed in M because it is the complement of the open set $C(a)$. Let $p \in M(a)$ be arbitrary. If $p \notin E([0,a] \times \partial M)$, then p has a neighborhood in $\operatorname{Int} M$ contained in $M(a)$ by the argument above. If $p \in E([0,a] \times \partial M)$, then $p = E(a,x)$ for some $x \in \partial M$, and C is a neighborhood of p in which $M(a) \cap C$ is the diffeomorphic image of $[a,1) \times \partial M$."

(1/30/14) **Page 223, proof of Theorem 9.26, last line of the first paragraph:** Change $0 \le t < a$ to $0 \le s < a$.

(1/30/14) **Page 225, Example 9.31:** At the end of the example, insert the sentence "If $n \ge 2$, then $M_1 \# M_2$ is connected."

(4/23/13) **Page 230, second paragraph:** "from Case" should be "from Case 1."

(6/4/14) **Page 246, Problem 9-11:** Delete the second sentence of the hint. [Because N is inward-pointing along ∂M, no integral curve that starts on ∂M can hit the boundary again, because the vector field would have to be tangent to ∂M or outward-pointing at the first such point.]

(7/2/14) **Page 264, paragraph above the subheading, first sentence:** "homomorphism" should be "homomorphisms."

(6/29/15) **Page 278, Example 11.13, third line:** Change "every coordinate frame" to "every coordinate coframe."

(4/17/15) **Page 333, first line:** Change $U \subseteq M$ to $V \subseteq M$.

(7/1/14) **Page 345, Problem 13-10:** In the last line of the problem statement, change $L_{\overline{g}}(\widetilde{\gamma}) > L_{\overline{g}}(\gamma)$ to $L_{\overline{g}}(\widetilde{\gamma}) \geq L_{\overline{g}}(\gamma)$, and delete the phrase "unless $\widetilde{\gamma}$ is a reparametrization of γ." [Because the definition of reparametrization that I'm using requires a diffeomorphism of the parameter domain, the original problem statement was not true.]

(12/18/12) **Page 355, proof of Lemma 14.10:** At the beginning of the proof, insert "Let $(E_1, \dots, E_n)$ be the basis for V dual to $\left(\varepsilon^i\right)$."

(12/18/12) **Page 356, Case 4, second line:** Should read "brings us back to Case 3."

(7/3/15) **Page 368, second paragraph:** At the end of the first sentence of the paragraph, insert "(see pp. 341–343)."

(9/17/14) **Page 371, three lines above (14.31):** Change that sentence to "The only terms in this sum that can possibly be nonzero are those for which J has no repeated indices and m is equal to one of the indices in J, say $m = j_p$."

(4/26/14) **Page 402, lines 2–3:** There should not be a paragraph break before "and."

(7/22/15) **Page 424, second displayed equation:** Change $\iota_S^*\beta(X)$ to $\iota_{\partial M}^*\beta(X)$.

(2/18/13) **Page 426, three lines below the section heading:** "cam" should be "can."

(2/11/15) **Page 430, Proposition 16.38(c):** This statement is wrong. Change it to "If F is smooth, then $F^*\mu$ is a continuous density on M; and if F is a local diffeomorphism, $F^*\mu$ is smooth."

(2/19/13) **Page 444, two lines below equation (17.4):** Change $T_{(q,s)}M$ to $T_{(q,s)}(M \times \mathbb{R})$.

(5/15/15) **Page 450, proof of Theorem 17.21, line 5:** Change $H^1_{\mathrm{dR}}(\mathbb{S}^n)$ to $H^1_{\mathrm{dR}}(\mathbb{S}^1)$.

(6/29/14) **Pages 455–456, Proof of Theorem 17.32:** The proof given in the book is incorrect, because the V_i's might not be connected, so Theorem 17.30 does not apply to them. Here's a corrected proof.

Lemma. *If M is a noncompact connected manifold, there is a countable, locally finite open cover $\{V_j\}_{j=1}^\infty$ of M such that each V_i is connected and precompact, and for each j, there exists $k > j$ such that $V_j \cap V_k \neq \varnothing$.*

Proof. Let $\{W_j\}_{j=1}^\infty$ be a countably infinite, locally finite cover of M by precompact, connected open sets (such a cover exists by Prop. 1.19 and Thm. 1.15). By successively deleting unneeded sets and renumbering, we can ensure that no W_j is contained in the union of the other W_i's.

Let $Y_1 = \bigcup_{i=2}^\infty W_i$. Because M is connected, each component of Y_1 meets W_1, and by local finiteness of $\{W_j\}$, there are only finitely many such components. Such a component is precompact in M if and only if it is a union of finitely many W_i's. Let V_1 be the union of W_1 together with all of the precompact components of Y_1, and let X_1 be the union of all W_i's not contained in V_1. Then V_1 is connected and precompact, and X_1 has no precompact components. Proceeding by induction, suppose we have defined connected, precompact open sets $V_1, \dots, V_m$ whose union contains $W_1 \cup \dots \cup W_m$, and such that the union X_m of all the W_i's not contained in some V_j has no precompact components. Let Y_{m+1} be the union of all W_i's other than W_{m+1} not contained in some V_j. Any precompact component of Y_{m+1} must meet W_{m+1}, because otherwise, it would be a precompact component of X_m. Let V_{m+1} be the union of W_{m+1} with all of the precompact components of Y_{m+1}. As before, V_{m+1} is precompact and connected, and the union X_{m+1} of the W_i's not contained in any of $V_1, \dots, V_{m+1}$ has no precompact components. Then by construction, for each j, the set $X_j = \bigcup_{i>j} V_i$ has no precompact components. If some V_j does not meet V_k for any $k > j$, then V_j itself is a precompact component of X_{j-1}, which is a contradiction. Thus for each j, there is some $k > j$ such that $V_j \cap V_k \neq \varnothing$. □

Proof of Theorem 17.32. Choose an orientation on M. Let $\{V_j\}_{j=1}^\infty$ be an open cover of M satisfying the conclusions of the preceding lemma. For each j, let $K(j)$ denote the least integer $k > j$ such that $V_j \cap V_k \neq \varnothing$, and let θ_j be an n-form compactly supported in $V_j \cap V_{K(j)}$ whose integral is 1. Let $\{\psi_j\}_{j=1}^\infty$ be a smooth partition of unity subordinate to $\{V_j\}_{j=1}^\infty$.

Now suppose ω is any n-form on M, and let $\omega_j = \psi_j \omega$ for each j. Let $c_1 = \int_{V_1} \omega_1$, so that $\omega_1 - c_1\theta_1$ is compactly supported in V_1 and has zero integral. It follows from Theorem 17.30 that there exists $\eta_1 \in \Omega_c^n(V_1)$

such that $d\eta_1 = \omega_1 - c_1\theta_1$. Suppose by induction that we have found $\eta_1, \dots, \eta_m$ and constants $c_1, \dots, c_m$ such that for each $j = 1, \dots, m$, $\eta_j \in \Omega^n_c(V_j)$ and

$$d\eta_j = \left(\omega_j + \sum_{i:K(i)=j} c_i\theta_i\right) - c_j\theta_j. \qquad (*)$$

Let

$$c_{j+1} = \int_{V_{j+1}} \left(\omega_{j+1} + \sum_{i:K(i)=j+1} c_i\theta_i\right).$$

Then by Theorem 17.30, there exists $\eta_{j+1} \in \Omega^n_c(V_{j+1})$ satisfying the analog of $(*)$ with j replaced by $j+1$. Set $\eta = \sum_{j=1}^{\infty} \eta_j$, with each η_j extended to be zero on $M \smallsetminus V_j$. By local finiteness, this is a smooth n-form on M. It satisfies

$$d\eta = d\omega + \sum_{j=1}^{\infty}\left(\sum_{i:K(i)=j} c_i\theta_i\right) - \sum_{j=1}^{\infty} c_j\theta_j.$$

Each term $c_i\theta_i$ appears exactly once in the first sum above, so the two sums cancel each other. □

(1/15/13) **Page 491, Example 19.1(c):** Delete the word "unit."

(5/22/15) **Page 492, line above Proposition 19.2:** Change "lie" to "Lie."

(12/17/15) **Page 492, proof of Proposition 19.2, fourth line:** Change "Given $p \in M$" to "Given $p \in U$."

(10/9/15) **Page 568, Example 22.9(a), first line:** The coordinates should be $\left(x^1, \dots, x^n, y^1, \dots, y^n\right)$. (The last coordinate is y^n, not x^n.)

(11/28/12) **Page 584, Exercise 22.29:** Part (b) should read

$$\text{(b)}\ \ T = \frac{\partial}{\partial z};$$

(8/14/14) **Page 584, paragraph above Theorem 22.33:** Change all occurrences of θ in this paragraph to ψ, to avoid confusion with the use of θ for a contact form elsewhere in this section.

(11/17/12) **Page 587, equation (22.27):** Change both occurrences of $\sigma(s)$ to $\sigma(x)$.

(9/22/15) **Page 608, Proposition A.41(a):** Insert the following phrase at the beginning of this statement: *With the exception of the word "closed" in part* (*d*).

(7/22/13) **Page 616, Proposition A.77(b), last line:** Change $\widetilde{f}(0)$ to $\widetilde{f}_e(0)$.

(12/2/15) **Page 666, just below the fifth display:** After the sentence ending "by our choice of δ and ε," insert "(If $t < t_0$, interchange t and t_0 in the second line above.)"

(12/2/15) **Page 667, proof of Theorem D.5, second paragraph:** In the first sentence of the paragraph, after "$\bar{J}_1 \subseteq J_0$," insert "and $\theta\left(\bar{J}_1 \times \{x_0\}\right) \subseteq U_0$." Then in the fourth line of that paragraph, change "$\bar{B}_{2c}(y) \subseteq U$" to "$\bar{B}_{2c}(y) \subseteq U_0$." [This is to ensure that $J_1 \times W$ will be contained in the domain of θ.]

(3/19/14) **Page 668, second line:** Change W to $\overline{W}$.

(3/19/14) **Page 668, paragraph below equation (D.10):** In the fourth line of the paragraph, change $\overline{W}$ to W, and in the fifth line, change W to $\overline{W}$.