Enhanced multivariable calculus

Wants to simultaneously generalize

① real valued functions of several variables,
② vector valued functions of one real variable,

to vector valued function of several real variables:

U open in $\mathbb{R}^n$ $\quad V$ open in $\mathbb{R}^m$

$f: U \to V$ s.t. coordinate functions have continuous derivatives of all orders (SMOOTH).

One obtains an $m \times n$ matrix

$$Df = \begin{pmatrix} \nabla f_1 \\ \vdots \\ \nabla f_m \end{pmatrix} = \left(\frac{\partial f_i}{\partial x_j}\right).$$

"Derivative of f"

<u>NOTE</u>: In many books and papers the coordinates are indicated by superscripts $f = (f^1, \ldots, f^m)$. Of course, these are **NOT** exponents!

<u>Simple but important example</u>: If A is an $m \times n$ matrix and $f(x) = Ax$, then $Df(x) = A$ for all x.

LINEAR APPROXIMATION PRINCIPLE

Let U, V, f be as above, $x \in U$, and choose δ so that $N_\delta(x; \mathbb{R}^n) \subseteq U$. Then for $|h| < \delta$ we have

$$f(x+h) = f(x) + Df(x)h + T_x(h)$$ ← trash term

where $\lim_{h \to 0} \frac{T_x(h)}{|h|} = 0$.

When $m = 1$ this is a standard result in second year calculus courses.

Here is a generalization of a basic result from multivariable calculus for the case $m = p = 1$:

CHAIN RULE:

Suppose we also have $g : W \longrightarrow U$ smooth where W is open in $\mathbb{R}^p$ and $x = g(y)$. Then

$f \circ g : W \longrightarrow V$ is also smooth and

$$Df\circ g(y) = Df(g(y)) \cdot Dg(y).$$

matrix product

<u>Definition</u> Take notation as above. A homeomorphism $f: U \to V$ is a <u>diffeomorphism</u> if both f and f^{-1} are smooth.

<u>Corollary of the definition</u> (INVERSE PRINCIPLE) If f is a diffeomorphism, then $Df(x)$ is invertible for all x and if $y = f(x)$ then

$$Df^{-1}(y) = Df(x)^{-1}$$

matrix inverse

<u>Sketch of proof</u> We know $D(\text{identity})(x) =$ identity by the special case on p. C-1. There for

$I = D[\text{identity}](x) = Df^{-1}(y) \cdot Df(x)$. Since we have $(f^{-1})^{-1} = f$ it also follows that $D[\text{identity}](y) = Df(x) \cdot Df^{-1}(y)$. Therefore $Df(x)$ and $Df^{-1}(y)$ are inverse to each other. ■

<u>Example</u> $f: \mathbb{R} \to \mathbb{R}$ $f(x) = x^3$. Then f is a smooth homeomorphism but not a diffeomorphism because $Df(0) = 0$. ■

On the otherhand, in first year calculus one learns that a smooth homeomorphism $f: (a,b) \longrightarrow (c,d)$ does have a smooth inverse if $f'(x)$ is neverzero. This has an important but nontrivial generalization to higher dimensions:

INVERSE FUNCTION THEOREM.
Let U, V be open in $\mathbb{R}^n$, let $x \in U$, and assume that $Df(x)$ is an isomorphism for a smooth map $f: U \longrightarrow V$. Then there are open neighborhoods U_0 of x and V_0 of $f(x)$ such that f maps U_0 diffeomorphically to V_0. ∎

EXAMPLE We can have $Df(x)$ invertible for all x but f is not a diffeomorphism. Consider the map

Polar Coordinates

$$\underset{\mathbb{R}^2 - \{0\}}{f(r,\theta)} \longrightarrow \underset{\mathbb{R}^2}{(r\cos\theta, r\sin\theta)}$$

Then Df is always invertible but $f(r, \theta+2\pi) = f(r,\theta)$ for all r and θ.

However, we do have the following:

COROLLARY. If f as above is a smooth homeomorphism and $Df(x)$ is invertible for all x, then f^{-1} is also smooth.

Sketch of proof Given $y \in V$ let $y = f(x)$, and take $U_0(x)$, $V_0(y)$ as in the Inverse Function Theorem. Then $f^{-1}|V_0(y)$ is smooth by that result. But the $V_0(y)$'s form an open covering of V, and a function is smooth $\Longleftrightarrow$ smooth on all sets in some open covering. Therefore f^{-1} is smooth.

There are proofs of the Inverse Function Theorem in Lee and many undergraduate textbooks on real analysis. A graduate student should be able to understand the argument at least passively (each step well-understood, but maybe not understood well enough to explain it to another student). Since the proof itself does not shed much light on the main topics of this course, reading through it is left to students.