Implicit Function Theorem

U open in $\mathbb{R}^n$, V open in $\mathbb{R}^m$, $f : U \times V \to \mathbb{R}^p$ smooth. The partial derivatives $D_1 f(u_0, v_0)$ and $D_2 f(u_0, v_0)$ are the derivatives of the composites

$$U \xrightarrow{\text{SLICE}} U \times V \xrightarrow{f} \mathbb{R}^p, \quad V \xrightarrow{\text{SLICE}} U \times V \xrightarrow{f} \mathbb{R}^p.$$

$$u \longrightarrow (u, v_0) \qquad\qquad v \longrightarrow (u_0, v)$$

IMPLICIT FUNCTION THEOREM. Let f be a smooth function as above, and assume that $p = m$ and $D_2 f(u_0, v_0)$ is an isomorphism whenever $f(u_0, v_0) = 0$. Then for each (u_0, v_0) there are open neighborhoods U' of u_0 and V' of V_0 and a smooth function $g : U' \to V'$ such that

$$f(u,v) = 0 \text{ in } U' \times V' \iff v = g(u).$$

Sketch of proof. By the corollaries to the Inverse function there is a diffeomorphism $h : U_0 \times V_0 \to U_1 \times V_1$ (with everything essentially contained in $U \times V$) such that $fh^{-1}(x,y) = y$. Take $g(x)$ to be $h^{-1}(x, 0)$. ∎