1. Smooth manifolds

Topological manifolds

$n \geq 0$ an integer

Def. A topological space X is a topological n-manifold if

① X is Hausdorff.

② Every point $x \in X$ has an open neighborhood homeomorphic to an open subset of $\mathbb{R}^n$.

Equivalently, we can assume

②' Every point $x \in X$ has an open neighborhood homeomorphic to some open n-disk

$$N_\varepsilon(0; \mathbb{R}^n) = \{ x \in \mathbb{R}^n \mid |x| \leq 1 \}.$$

SIMPLE OBSERVATIONS.

1. If X is a topological manifold, it is locally compact (and hence T_3). This is true because $\overline{N_{\varepsilon/2}(0)} \subseteq N_\varepsilon(0)$ is compact in $\mathbb{R}^n$.

2. There is at most one n such that X is a topological n-manifold. By Invariance of dimension, if $U \subseteq \mathbb{R}^m$ and $V \subseteq \mathbb{R}^n$ are homeo. open sets, then $m=n$. — Call n the <u>dimension</u> of X.

3. A topological 0-manifold is a set with the discrete topology.

4. The connected components of a topological n-manifold are also the arc components, and each such subset is both open and closed.

5. A topological n-manifold is metrizable $\Longleftrightarrow$ each component is metrizable, which in turn is equivalent to saying that each connected component is Second Countable. The second part is nontrivial but accessible to an entry level graduate student (at least for passively understanding the proof).

EXAMPLES:

1. Open subsets in $\mathbb{R}^n$.

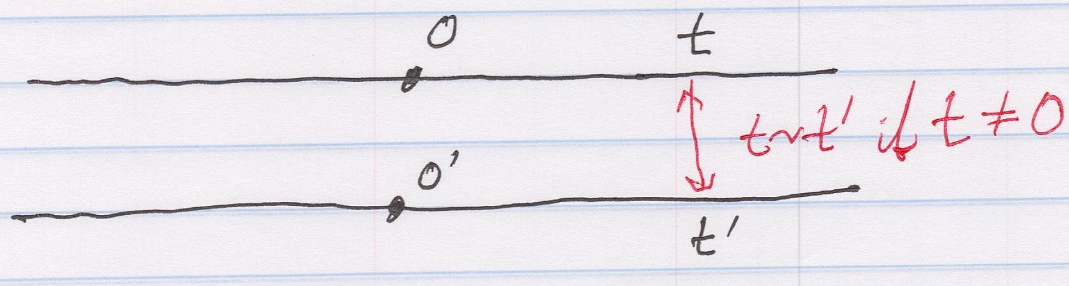
2. The sphere $S^n = \{x \in \mathbb{R}^{n+1} \mid |x| = 1\}$. By stereographic projection, for each $p \in S^n$ the complement $S^n - \{p\}$ is homeo to $\mathbb{R}^n$.

3. The n-torus $T^n = S^1 \times \dots \times S^1$ (n factors)
More generally,

<u>FACT</u> Let $X_1, \dots, X_k$ s.t. X_j is a topological n_j-manifold. Then $X_1 \times \dots \times X_k$ is a topological $(n_1 + \dots + n_k)$-manifold.

<u>And here are some nonexamples</u>

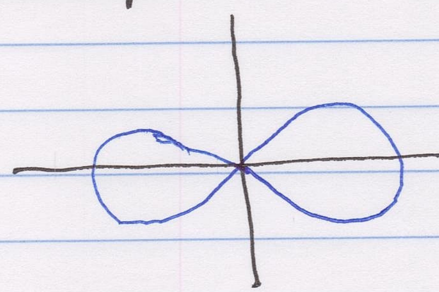
4. A space can satisfy ② but not be Hausdorff. Look at the Forked Line $\mathbb{R} \amalg \mathbb{R}$ mod the quotient relation identifying the two copies of $\mathbb{R} - \{0\}$ via the identity:



Then the quotient space $\longleftrightarrow \mathbb{R}-\{0\} \cup \{0', 0''\}$
where $(\mathbb{R}-\{0\}) \cup \{0'\} \cong \mathbb{R}$
$(\mathbb{R}-\{0\}) \cup \{0''\} \cong \mathbb{R}$ but every pair of open neighborhoods for $0'$ and $0''$ meet in some set of the form $(-\varepsilon, 0) \cup (0, \varepsilon)$.

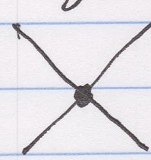
5. In longline.pdf there is a connected topological 1-manifold which is not second countable. This can be propagated to yield examples in all positive dimensions.

6. A figure 8 curve in the plane is not a topological manifold.



Explicit example:

$$\left.\begin{aligned} x(t) &= \cos t \\ y(t) &= \sin 2t \end{aligned}\right\} 0 \le t \le 2\pi$$

Near the origin this has an open neighborhood N of the form , so $N - \{(0,0)\}$ has 4 components. Furthermore if $(0,0) \in U$ open $\subseteq X$,

then $U - \{(0,0)\}$ always has ≥ 4 components.
In contrast, we have

PROPOSITION. Let X be a topological n-manifold, and let $p \in X$. Then p has an open neighborhood base $\{U_\alpha\}$ of the following type:

(I) If $n = 1$, each $U_\alpha - \{p\}$ has two components.

(II) If $n \geq 2$, each $U_\alpha - \{p\}$ has one component.

SKETCH OF PROOF Since p has an open neighborhood homeomorphic to the disk $N_\varepsilon(0; \mathbb{R}^n)$, it suffices to consider the case where X is that disk and $p = 0$, where we have a neighborhood base of the form $\{N_\delta(0; \mathbb{R}^n)\}$ where $0 < \delta < \varepsilon$.

But for each of these families we know that

$$N_\delta(0; \mathbb{R}^n) - \{0\} \text{ has } \begin{Bmatrix} 2 \\ 1 \end{Bmatrix} \text{ components if } \begin{Bmatrix} n = 1 \\ n \geq 1 \end{Bmatrix}.$$

Since we have seen that $(0,0) \in X =$ Figure 8 has no such nbhd bases, X is not a topological manifold. ∎

We should also note that there are some second countable topological manifolds which are not homeomorphic to open subsets in $\mathbb{R}^n$:

PROPOSITION If X is a nonempty compact topological n-manifold, then X is not homeomorphic to an open subset of $\mathbb{R}^n$. $(n \geq 1)$

In particular, we cannot flatten S^2 into an open subset of $\mathbb{R}^2$ without puncturing or tearing it (which is clear from physical experience).

PROOF. Say $X \cong U$ open in $\mathbb{R}^n$. Then U is a compact open subset of $\mathbb{R}^n$; Since a compact subset of a Hausdorff space is closed, we see that U is closed in $\mathbb{R}^n$, and since $U \neq \emptyset$ and $\mathbb{R}^n$ is connected we must have $U = \mathbb{R}^n$. Therefore $\mathbb{R}^n$ is also compact; since this is false for $n \geq 1$, there is no open subset U of $\mathbb{R}^n$ which is homeomorphic to X. ∎

A SIMPLE BUT USEFUL FACT.

Let $x \in X$ where X is a topological n-manifold. Then x has a neighborhood base consisting of sets $\{U_\alpha\}$, each of which is homeomorphic to an open subset in $\mathbb{R}^n$.

SKETCH OF PROOF. An open subset of a topological manifold is a topological manifold [If $W \subseteq X$ open and $x \in W$, let $x \in U$ open with $U \cong$ open set in $\mathbb{R}^n$. Then $W \cap U \cong$ open set in $\mathbb{R}^n$ and $x \in W \cap U$. Of course X Hausdorff $\Rightarrow W$ + $W \cap U$ are too.]

We can now apply equivalent criterion 2''' in the "definition" of a topological manifold. ■

Fundamental groups and covering spaces.

Let $p: E \to B$ be a covering space projection where E and B are Hausdorff. Then E is a topological n-manifold $\Leftrightarrow$ B is.

SKETCH OF PROOF. ($\Rightarrow$). If $e \in E$, let $e \in U$ open so that $p[U] \subseteq B$ is evenly covered. Then $\exists$ V open in E s.t. $e \in V \subseteq U$ and $V \cong$ open set in $\mathbb{R}^n$. — Given $b \in B$ choose e so that $p(e) = b$ and choose U, V as above. Then $b \in p[V]$ and the latter is an open neighborhood which is homeomorphic to an open set in $\mathbb{R}^n$.

($\Leftarrow$) Let $e \in E$ and choose W open in B such that $p(e) \in W$ and W is evenly covered. Now let $\Omega \subseteq W$ be an open neighborhood of $p(e)$ which is homeomorphic to an open subset in $\mathbb{R}^n$. Then $p^{-1}[\Omega]$ is a disjoint union of copies of Ω and e lies in exactly one of these copies. ∎

Universal coverings. Every point p of a topological n-manifold has a neighborhood base of contractible open sets, for it has a base homeomorphic to $\{N_\delta(0;\mathbb{R}^n)\}$, δ small, with $0 \Longleftrightarrow p$.

Therefore every connected topological n-manifold has a universal (simply connected) covering space which is also a topological n-manifold.

THEOREM If X is a connected topological n-manifold and $x \in X$, then $\pi_1(X, x)$ is a countable group

(see Lee for a proof)

There is also a partial converse.

If G is a countable group and $n \geq 4$, then there is a topological n-manifold X which is connected and has $\pi_1(X, x) \cong G$.

Default hypothesis From now on, all manifolds are second countable (actually, also for everything on this page).