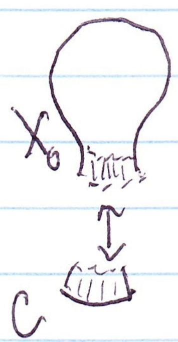
# Manifolds with boundary

In the course of studying smooth manifolds, one often encounters objects like $\mathbb{R}^n_+ = [0,\infty) \times \mathbb{R}^{n-1}$ (half-space) and the disk $D^n = \{x \in \mathbb{R}^n \mid |x| = 1\}$ which have a boundary of some sort. We shall cover this very briefly with definitions that are equivalent to those in Chapter 1 of Lee.




**Def.** A topological n-manifold with boundary is a space* $X$ with a decomposition into two open subsets

$X_0$ (the interior), $C$ (an open boundary collar)

such that the following hold:

1. $X_0$ is a topological $n$-manifold
2. $C$ is homeomorphic to $Y \times [0,\infty)$ for some $(n-1)$-manifold $Y$ (write $Y = \partial X$).
3. $X_0 \cap C \longleftrightarrow Y \times (0,\infty)$

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* always Hausdorff and 2nd countable unless stated otherwise.

EXAMPLES.

0. A topological $n$-manifold $X$, with $X = X_0$, $C = Y = \emptyset$. (no boundary)

1. A product $Y \times (0, \infty)$ where $Y$ is a topological $(n-1)$-manifold, with $X_0 = \emptyset$, $C = X$. (includes $\mathbb{R}^n_+$).

2. The disk $D^n$, where $X_0 = \{ |x| < 1 \}$

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$Y = S_{n-1}$, $C = \{ |x| > \frac{1}{2} \}$ and we have

$$C \cong S^{n-1} \times (\tfrac{1}{2}, 1] \cong S^{n-1} \times [0, \infty)$$

$$x \longrightarrow \left( \frac{x}{|x|}, |x| \right) \longrightarrow \left( \frac{x}{|x|}, h(|x|) \right)$$

where $h : (\frac{1}{2}, 1] \cong [0, \infty)$.

Def. A **smooth structure** on a top. $n$-mfld with boundary consists of smooth structures on $X_0$ and $Y$ s.t. some collar $C$ is diffeomorphic to $Y \times (0, \infty)$.

A mapping of manifolds with boundary is smooth if it extends to a smooth map from $X_0 \cup_C (Y \times \mathbb{R})$ to $X_0' \cup_{C'} (Y' \times \mathbb{R})$, where the latter have the standard smooth structures.

## EXAMPLES

All the preceding examples have "obvious" smooth structures.

4. If $X$ has a smooth structure, then the inclusion $\partial X \subseteq X$ is smooth.

See pp. 27-29 of Lee for a more abstract approach. The equivalence between Lee's definition and ours in the smooth case is given by the smooth Collar Neighborhood Theorem on pp. 222-224 of Lee. In the topological setting, there is a similar topological Collar Neighborhood Theorem due to M. Brown (mainly) and E. Michael.

Here are references:

M. Brown, in Topology of 3-manifolds and related topics (pp. 83-91), Prentice-Hall, 1962.

R. Connelly, Proc. Amer. Math. Soc. 27 (1971), 180-182. (Simplified proof of Brown's result).

E. Michael, General Topology and its Relations ... (Prague, 1961), pp. 270-271. Academic Press, 1962. (Fills in a small gap in Brown's proof.)