Constructions on smooth manifolds

PRODUCTS. We shall only do the case of a product with two factors. Products with more factors can be defined recursively by $X_1 \times \ldots \times X_m = (X_1 \times \ldots \times X_{m-1}) \times X_m$.

STRAIGHTFORWARD OBSERVATION. If U_1, V_1, U_2, V_2 are open in coord spaces and $f_i : U_i \to V_i$ is smooth, then so is $h = f_1 \times f_2 : U_1 \times U_2 \longrightarrow V_1 \times V_2$.

In fact, we have $Dh = \begin{pmatrix} Df_1 & 0 \\ 0 & Df_2 \end{pmatrix}$ in block form.

COROLLARY. If $(M, \mathcal{A})$ and $(N, \mathcal{B})$ are smooth manifolds with maximal atlases $\{h_\alpha : W_\alpha \to U_\alpha\}$ and $\{k_\beta : V_\beta \to \Omega_\beta\}$, then the mappings $h_\alpha \times k_\beta$ define a smooth atlas for $M \times N$.

Def. The $\boxed{\text{product atlas}}$ $\mathcal{A} \prod \mathcal{B}$ (uppercase Pi) is the maximal atlas determined by this atlas for $M \times N$.

FUNDAMENTAL PROPERTY OF PRODUCTS.

Suppose we are given smooth manifolds $(M, \mathcal{A})$, $(N, \mathcal{B})$ and $(P, \mathcal{C})$, and suppose that $f : P \longrightarrow M \times N$ is continuous. Let

$\pi_M : M \times N \longrightarrow M$ and $\pi_N : M \times N \longrightarrow N$ be the respective coordinate projections. Then f defines a smooth map $(P, \mathcal{C}) \longrightarrow (M \times N, \mathcal{A} \prod \mathcal{B})$ $\Longleftrightarrow$ the coordinat maps $\pi_M \circ f$ and $\pi_N \circ f$ are smooth.

<u>Proof.</u> ($\Longrightarrow$) Since composites of smooth maps are smooth, it will suffice to show that π_M + π_N are smooth. These follow because the latter are given locally by the coordinate projections $U_\alpha \times \mathcal{V}_\beta \rightarrow V_\alpha$ and $U_\alpha \times \mathcal{V}_\beta \rightarrow \mathcal{V}_\beta$, and the latter are smooth because they are linear mappings. ∎

($\Longleftarrow$) By continuity there are smooth charts

Let $p \in P$.

$$\ell_\gamma : X_\gamma \longrightarrow Q_\gamma \quad \text{in } \mathcal{C} \qquad p \in X_\gamma$$
$$h_\alpha : W_\alpha \longrightarrow U_\alpha \quad \text{in } \mathcal{Ol}$$
$$k_\beta : \Omega_\beta \longrightarrow V_\beta \quad \text{in } \mathcal{B}$$

such that $f[X_\gamma] \subseteq W_\alpha \times \Omega_\beta$ and "$(h_\alpha \times k_\beta) \circ f \circ \ell_\gamma^{-1}$" is smooth.

By construction the projections of the latter onto U_α and V_β are equal to

"$h_\alpha \circ (f \pi_M \circ f) \circ \ell_\gamma^{-1}$" and

"$k_\beta \circ (\pi_N \circ f) \circ \ell_\gamma^{-1}$" respectively,

and these are smooth because $\pi_M \circ f$ and $\pi_N \circ f$ are both smooth. Since "$(k_\alpha \times k_\beta) \circ f \circ \ell_\gamma^{-1}$" is smooth $\Longleftrightarrow$ its coordinate projections are smooth (it's a map from an open set in $\mathbb{R}^p$ to an open set in $\mathbb{R}^{m+n}$), it follows that this local map is smooth. ∎

<u>COVERING SPACE PROJECTIONS.</u> Let $(M, \mathcal{Ol})$ be a smooth manifold, and let $p : E \longrightarrow M$ be a (Hausdorff) covering space projection.

Let $\mathfrak{A}' \subseteq \mathfrak{A}$ be the subatlas of charts so that the domain is evenly covered. Given a chart $h_\alpha : W_\alpha \to U_\alpha$ where W_α is evenly covered, write $p^{-1}[W_\alpha] \cong \coprod_t W_{\alpha,t}$ such that p maps each $W_{\alpha,t}$ homeomorphically onto W_α.
Consider the coordinate charts $h_{\alpha,t}$

$$W_{\alpha,t} \xrightarrow{\text{“}p\text{”}} W_\alpha \xrightarrow{h_\alpha} U_\alpha .$$

We $\boxed{\text{claim}}$ these form a smooth atlas for E.
This follows immediately from the "identity"
$$\text{“}h_{\beta,s} \circ h_{\alpha,t}^{-1}\text{”} = \text{“}h_\beta \circ h_\alpha^{-1}\text{”}.$$ Summarizing,

THEOREM. Let E and $(M, \mathfrak{A})$ be as above. Then there is a smooth structure $(E, p^*\mathfrak{A})$ such that for each evenly covered smooth chart $h_\alpha : W_\alpha \to U_\alpha$ in M the map p sends each sheet over W_α diffeomorphically onto W_α.

<u>Proof</u>. The only thing left to prove is the assertion that p maps each $W_{\alpha,t}$ diffeomorphically onto W_α. By construction we know that "$h_\alpha \circ p \circ h_{\alpha,t}^{-1}$" is definable, and it also follows that this map is the identity. Therefore p maps $W_{\alpha,t}$ diffeomorphically to W_α, as required. ∎

Here is another basic result on covering space atlases.

<u>THEOREM.</u> Let $(M, \mathcal{O})$ be a smooth manifold, and let $(E, p^*\mathcal{O})$ be the associated smooth structure for the (Hausdorff) covering space projection $p: E \longrightarrow B$. If $T: E \longrightarrow E$ is a covering space projection,* then T is smooth (hence a diffeomorphism). (*: $p \circ T = p$)

<u>Proof</u>. Locally T maps a set $W_{\alpha,t}$ to $W_{\alpha,s}$ for some s, and therefore it is defined locally by

$$U_\alpha \xrightarrow{\text{``}h_{\alpha;t}^{-1}\text{''}} W_{\alpha;t} \xrightarrow{\text{``}T\text{''}} W_{\alpha,s} \xrightarrow{h_{\alpha,s}} U_\alpha$$

and by construction this map is the identity. ∎

Finally, we have the following.

THEOREM. Let $(M, \mathcal{O})$ and $p : E \to M$ be as above, and suppose that $f : M \to N$ is continuous for some smooth manifold $(N, \mathcal{B})$. Then f is smooth $\Longleftrightarrow$ $f \circ p$ is smooth.

<u>Proof</u> ($\Rightarrow$) Both f and p are smooth, and the composite of two smooth mappings is smooth.

($\Leftarrow$) Given $y \in E$ choose smooth coordinate charts $W_\alpha \xrightarrow{h_\alpha} U_\alpha$ at $p(y)$ and $\Omega_\beta \xrightarrow{k_\beta} V_\beta$ such that W_α is evenly covered and $f[W_\alpha] \subseteq \Omega_\beta$. If $W_{\alpha,t} \xrightarrow{H_{\alpha,t}} U_\alpha$ is the corresponding chart at $y \in E$, the assumptions imply that "$k_\beta \circ (f \circ p) \circ H_{\alpha,t}^{-1}$" is smooth.

By construction the latter is equal to "$k_\beta \circ f \circ h_\alpha^{-1}$", and hence this local map is smooth; but this proves that f is smooth. ■