Partitions of unity

TOPOLOGICAL PRELIMINARIES

PROPOSITION. Let X be a locally compact Hausdorff space which is second countable. Then the following hold:

① X has a countable base of open subsets with compact closures.

② There is an increasing sequence of compact subsets $K_1 \subseteq K_2 \subseteq \ldots$ such that $K_n \subseteq \mathrm{Int}\, K_{n+1}$ for all n and $X = \bigcup_n K_n$.

Proof ① Let $\mathcal{B}$ be a countable base, and let $\mathcal{B}' \subseteq \mathcal{B}$ consist of all open subsets in $\mathcal{B}$ with compact closures. If U is an open subset of X, then by the local compactness and Hausdorff conditions we have $U = \cup V_\alpha$ where $\overline{V_\alpha}$ is compact. Since $W \subseteq V_\alpha \Rightarrow \overline{W} \subseteq \overline{V_\alpha}$, if we write V_α as a union of open subsets in $\mathcal{B}$, these subsets actually lie in $\mathcal{B}'$. Therefore U is a union of (unions of) sets in $\mathcal{B}' \subseteq \mathcal{B}$, so $\mathcal{B}'$ is a base for X. ■

(2) Let $\mathcal{B}'$ be as in (1) and write the sets in sequence $W_1, W_2, \ldots$. If X is compact there is nothing to prove, so if necessary we may assume X is noncompact. We shall construct K_n recursively so that $K_n \supseteq W_n$ for all n. Set $K_1 = \overline{W_1}$. Suppose now that we have K_n for some $n \geq 1$. By compactness, K_n is contained in some finite union $W_{\alpha_1} \cup \ldots \cup W_{\alpha_{M(n)}}$; take K_{n+1} to be the closure of $W_{n+1} \cup W_{\alpha_1} \cup \ldots \cup W_{\alpha_{M(n)}}$. Then K_{n+1} is the union of the compact closures $\overline{W_{n+1}} \cup \overline{W_{\alpha_1}} \cup \ldots \cup \overline{W_{\alpha_{M(n)}}}$ and hence is compact. Furthermore, $K_n \subseteq W_{\alpha_1} \cup \ldots \cup W_{\alpha_{M(n)}} \subseteq \mathrm{Int}\, K_{n+1}$ by construction. ∎

$\boxed{\text{MAIN EXAMPLE}}$ $X = \mathbb{R}^k$, $K_n = \{ |v| \leq n \}$.

(In the file annuli.pdf the set K_n corresponds to $A_0 \cup \ldots \cup A_n$.)

More generally, if $A_n = K_n - \mathrm{Int}\, K_{n-1}$ and

X loc. comp. T_2, 2 count.

$V_n = \mathrm{Int}\, K_{n+1} - K_{n-2}$

empty if $n-2 \leq 0$

then A_n is compact, V_n is open, and we have $A_n \subseteq V_n$

$$V_i \cap V_j = \emptyset \text{ if } |i-j| \geq 3.$$

(look at the picture in the previously cited file).

<u>Definition</u> A family $\mathcal{S} = \{S_\alpha\}$ of subsets in a topological space X is <u>locally finite</u> if for each $x \in X$ there is an open neighborhood U of x such that $U \cap S_\alpha = \emptyset$ for all but finitely many S_α.

EXAMPLE. If X is as above, then $\mathfrak{V} = \{V_0, V_1, \ldots\}$ is a locally finite open covering of X. In fact, if $y \in K_n$, then

$$\text{Int } K_{n+1} \cap V_j = \emptyset \text{ for } j \geq n+3.$$

<u>THEOREM.</u> Let M^n be a second countable topological n-manifold. Then M^n has a countable locally finite open covering by coordinate charts $h_\alpha : W_\alpha \longrightarrow N_3(0; \mathbb{R}^n)$ such that the shrunken charts $h_\alpha^{-1}[N_1(0;\mathbb{R}^n)]$ still cover M^n.

<u>Note.</u> Every such manifold actually has an open covering by $\boxed{n+1}$ subsets which are homeomorphic to an open n-disk (E. Luft). This is the best possible general statement, for $n+1$ open n-disks are required to cover $T^n = \underbrace{S^1 \times \ldots \times S^1}_{n}$ (follows from results of Lyusternik and Schnirel'man).

<u>ADDENDUM TO THEOREM.</u> If $\mathcal{U}$ is an open covering of M^n, we can choose the W_α so that $\mathfrak{W} = \{W_\alpha\}$ refines $\mathcal{U}$. Furthermore, if M^n is compact we can take $\mathfrak{W}$ to be finite.

(The final sentence is easy to check!)

<u>Proof.</u> First of all, for each $x \in M^n$ there is a coordinate chart $h_x : W_x \longrightarrow N_3(0; \mathbb{R}^n)$ such that each W_x is contained in some U_α from $\mathcal{U}$. Furthermore, if $\{V_0, V_1, \ldots\}$ is the locally finite open covering described on the preceding two pages, we can also choose the charts so that each $W_x \subseteq$ some V_j.

<u>let</u> $\mathcal{B}_x = h_x^{-1}[N_1(0;\mathbb{R}^n)]$, so that $\mathcal{B}_x$ is compact and is contained in W_x. Take K_j, A_j and V_j as above, and for each j let $\mathcal{S}_j$ be a finite subcollection of $\{h_x\}$ such that the corresponding $\mathcal{B}_x$'s cover A_j. By construction each open subset in $\mathcal{S}_j$ is contained in some U_α. If $\mathcal{S} = \cup \mathcal{S}_j$, we claim $\mathcal{S}$ is locally finite (it's clearly countable). Given $x \in M$, choose q so that $x \in V_q$ and note that $x \notin V_r$ if $|r-q| \geq 3$. By construction each set in $\mathcal{S}_j$ is contained in V_j and hence the only subsets of $\mathcal{S}$ which intersect V_q nontrivially are those in $\mathcal{S}_r$ for $|r-q| \leq 2$. Each collection $\mathcal{S}_j$ is finite, so it follows that only finitely many open sets in $\mathcal{S}$ meet V_q nontrivially. Therefore $\mathcal{S}$ is locally finite. ∎

EXISTENCE OF PARTITIONS OF UNITY.

Let M^n be a second countable topological manifold, and let $\{h_\alpha : W_\alpha \longrightarrow N_3(0;\mathbb{R}^n)\}$ be a locally finite open covering as in the preceding result. Then there is a family of continuous functions $\varphi_\alpha : M^n \longrightarrow [0,1]$ such that $\varphi_\alpha = 0$ off $h_\alpha^{-1}[N_2(0;\mathbb{R}^n)]$ and

$$\sum \varphi_\alpha = 1$$

(i.e., a partition of unity).

NOTE. There are no convergence problems. By local finiteness each $p \in M$ has an open neighborhood $\mathcal{O}$ such that $\mathcal{O} \cap W_\alpha \neq \phi$ for at most finitely many α, and $\varphi_\alpha | \mathcal{O} = 0$ if $\mathcal{O} \cap W_\alpha = \phi$. Therefore $\sum \varphi_\alpha | \mathcal{O}$ collapses to a finite sum.

Proof Let $\overline{\varphi} : [0,\infty) \longrightarrow [0,1]$ be a continuous function which is 1 on $[0,1]$, maps $[1,2]$ to $[0,1]$ linearly by $t \longrightarrow 2-t$, and is 0 on $[2,\infty)$. Define φ_α on W_α by $\overline{\varphi}(|h_\alpha(y)|)$.

Since $\overline{\varphi}_\alpha = 0$ off the compact ($\Rightarrow$ closed) subset $h_\alpha^{-1}[\overline{N_2(0; \mathbb{R}^n)}]$, it follows that we can extend $\overline{\varphi}_\alpha$ to a continuous function on M^n by setting it equal to 0 on $M^n - h_\alpha^{-1}[\overline{N_2(0; \mathbb{R}^n)}]$.

<u>**CLAIM:**</u> $\sum \overline{\varphi}_\alpha > 0$.

This is true because each $p \in M$ lies in some subset $h_{\alpha_0}^{-1}[N_1(0; \mathbb{R}^m)]$ and $\overline{\varphi}_{\alpha_0} = 1$ on this subset.

If we now let $\varphi_\alpha = \dfrac{\overline{\varphi}_\alpha}{\sum_\beta \overline{\varphi}_\beta}$, then φ_α has the required properties. ∎