Partitions of unity continued

APPLICATIONS.

THEOREM. Let M be a second countable topological manifold, and let U be an open neighborhood of $M \times \{0\}$ in $M \times [0,1)$. Then there is a continuous function $h: M \longrightarrow (0,1)$ such that if $t \leq h(x)$, then $(x,t) \in U$.

NOTE. The open subset $(x,t) \in M \times [0,1)$ s.t. $t < h(x)$ is homeomorphic to $M \times [0,1)$ by the map sending (x,t) to $\left(x, \frac{t}{h(x)}\right)$, so we can say "$M \times \{0\}$ has a neighborhood base of open collar neighborhoods."

Proof. There is an open covering $\mathcal{U} = \{U_\beta\}$ of M such that for each β there is some $\varepsilon_\beta > 0$ with $U_\beta \times [0, \varepsilon_\beta) \subseteq U$. Take an open locally finite refinement $\mathcal{W} = \{W_\alpha\}$ as before, and let $\{\varphi_\alpha\}$ be an associated partition of unity. Take $h = \sum \varepsilon_{\beta(\alpha)} \varphi_\alpha$, where $W_\alpha \subseteq U_{\beta(\alpha)}$. Note that $h(x) > 0$, all x.

CLAIM: h has the required properties.

Let $x \in M$. By local finiteness there are only finitely many α's so that $x \in W_\alpha$. If ε^* is the maximum of the corresponding $\varepsilon_{\beta(\alpha)}$'s then $\{x\} \times [0, \varepsilon^*) \subseteq U$. By construction $h(x) = \sum \varepsilon_{\beta(\alpha)}(x) \varphi_\alpha(x) \leq \varepsilon^*$, and hence

$$t < h(x) \Rightarrow (x, t) \in U. \blacksquare$$

THEOREM. If M^n is a compact topological manifold, then M^n is homeomorphic to a (closed) subset of some $\mathbb{R}^k$.

Proof. There is a finite open covering $\{h_\alpha : W_\alpha \longrightarrow N_3(0; \mathbb{R}^n)\}$ such that the sets $\Omega_\alpha = h_\alpha^{-1}[N_1(0; \mathbb{R}^n)]$ still cover M^n.

Take the associated partition of unity $\{\varphi_\alpha\}$, and define continuous mappings $f_\alpha : M^n \longrightarrow \mathbb{R}^{n+1}$ by

$$f_\alpha(x) = \begin{cases} (\underbrace{\varphi_\alpha(x)\cdot h(x)}_{\in \mathbb{R}^n}, \underbrace{\varphi_\alpha(x)}_{\in \mathbb{R}}), & x \in W_\alpha \\ 0, & x \notin h_\alpha^{-1}[\overline{N_2(0,\mathbb{R}^n)}] \end{cases}$$

(check these are consistent on the overlapping pieces).

Now take the indexing set to be $\{1, \ldots, q\}$ and define $g : M^n \longrightarrow \mathbb{R}^{(n+1)q}$ such that the projection onto the jth factor is f_j.

Since M^n is compact Hausdorff, it suffices to show that f is 1-1. Suppose that $f(y) = f(z)$. Then for all j we have

$$\varphi_j(y) = \varphi_j(z)$$
$$\text{“}\varphi_j(y)\,h_j(y) = \varphi_j(z)\,h_j(z)\text{”}.$$

Note this $= 0$ off W_j

There must be some r such that $\varphi_r(y) = \varphi_r(z) > 0$, and this means that both y and z lie in W_r. Furthermore, we must have $h_r(y) = h_r(z)$ in this case. Since h_r is 1-1, it follows that $y = z$.

The same conclusion also holds for noncompact 2nd countable manifolds, but a more refined argument is needed.

SMOOTH PARTITIONS OF UNITY.

If M has a smooth structure, one can modify the construction to get a smooth partition of unity.

In the proofs of the theorem and complement stated on pp. 2C-3 and 2C-4, if $(M, \mathfrak{A})$ is a smooth manifold then we can choose the coordinate charts to be smooth charts. However, in the existence proof the function φ is not smooth, so we need the following result to use the same sort of argument:

BUMP FUNCTION THEOREM. There is a C^∞ function $\overline{\varphi}: \mathbb{R} \to [0,1]$ such that $\overline{\varphi} = 1$ for $t \leq 1$, $\overline{\varphi}$ is strictly decreasing on $[1,2]$ and $\overline{\varphi} = 0$ for $t \geq 2$.

PROOF OF THE BUMP FUNCTION THEOREM.

Step 1 Construct a C^∞ function $f: \mathbb{R} \to \mathbb{R}$ s.t. $f(t) = 0$ for $t \leq 0$ and $f(t) > 0$ for $t > 0$.

Lee shows that if we set $f(t) = e^{-1/t}$ for $t > 0$, then the resulting function f has this property because $\frac{d^k f}{dt^k}(0+) = 0$ for all $k \geq 0$ (in fact, Lee outlines how one proves this using L'Hospital's Rule — actually, the latter is probably due to Johann Bernoulli, but whatever...)

Step 2 If $g(t) = f(t) \cdot f(1-t)$ then g is C^∞ with $g = 0$ for $t \notin (0,1)$ and $g > 0$ on $(0,1)$.

Step 3. If $A = \int_0^1 g(u)\,du$ (positive!)

and $c(t) = \frac{1}{A}\int_0^t g(s)\,ds$, then

$c(t) = 0$ for $t \leq 0$, $c(t)$ is strictly increasing

for $t \in [0,1]$ and $c(t) = 1$ for $t \geq 1$.

Step 4. We can now take

$$h(t) = c(2-t). \blacksquare$$

See also the application on pages 44 through 47 of Lee (note that an exhaustion function which is bounded from below is a proper map).