3. Tangent spaces

We need a global framework for studying the differentiation of smooth maps on smooth manifolds. This is given by the tangent space of a smooth manifold, called $T(M)$. In fact this construction extends to a covariant functor on smooth manifolds and smooth mappings, with values in the same category.

Although our construction differs from that of Lee's book, the two constructions are equivalent and satisfy the following properties (which uniquely determine $T(M)$ and $T(f)$ up to equivalence):

<u>Local description</u> If $U \subseteq \mathbb{R}^n$ is open, then $T(U)$ is naturally equivalent to $U \times \mathbb{R}^n$ with the following extra structure:

Each $\{u\} \times \mathbb{R}^n$ is viewed as an n-dimensional vector space, and there is a parametrized vector space structure $\left(\tau_U : U \times \mathbb{R}^n \xrightarrow[\text{projection}]{} U\right)$

all (ξ_1, ξ_2) with $\tau(\xi_1) = \tau(\xi_2)$

$$\begin{array}{ccc}
 & \shortparallel & \\
\alpha : T(U) \times_U T(U) & \longrightarrow & T(U) \\
\cong & & \cong \\
U \times \mathbb{R}^n \times \mathbb{R}^n & & U \times \mathbb{R}^n \\
(u, A, B) & \longrightarrow & (u, A+B)
\end{array}$$

$$\begin{array}{ccc}
\mu : \mathbb{R} \times T(U) & \longrightarrow & T(U) \\
\cong & & \cong \\
\mathbb{R} \times U \times \mathbb{R}^n & & U \times \mathbb{R}^n \\
(t, u, A) & \longrightarrow & (u, tA).
\end{array}$$

Say $T_u(U) \cong \{u\} \times \mathbb{R}^n$ is the tangent space at u (so tangent vectors have a "point of application" as in physics).

Furthermore, if

$$\begin{array}{ccc}
f : U & \longrightarrow & V \\
\cap & & \cap \\
\mathbb{R}^n & & \mathbb{R}^m
\end{array} \quad \text{is smooth}$$

then $T(f)(u, A) = (f(u), Df(u)A)$.

Hence $T_u(f) : T_u(U) \longrightarrow T_{f(u)}(V)$

is a linear map of tangent spaces. In particular,

$$\begin{tikzcd}
T(M) \arrow[r, "T(f)"] \arrow[d, "\tau_M"'] \arrow[dr, phantom, "="] & T(N) \arrow[d, "\tau_N"] \\
M \arrow[r] & N
\end{tikzcd} \qquad \begin{pmatrix} M = U \\ N = V \end{pmatrix}.$$

The structure we need is a special case of a <u>smooth vector bundle</u>, whose global structure is given by $\tau_M : T(M) \longrightarrow M^n$ and maps $T(f)$ which satisfy the compatibility condition at the bottom of the preceding page. We want $T(M)$ to be a smoothly parametrized family of n-dimensional vector spaces. More precisely, if $T(M) \times_M T(M)$ is the set of all $(A, B) \in T(M) \times T(M)$ such that $\tau_M(A) = \tau_M(B)$, we want a smooth structure on $T(M) \times_M T(M)$ and smooth maps

$$\begin{tikzcd} T(M) \times_M T(M) \arrow[r, "\alpha"] \arrow[d, "\tau_M^{(2)}"] & T(M) \arrow[d, "\tau_M"] \\ T(M) \arrow[r, "="] & T(M) \end{tikzcd}$$

with the compatibility condition

$\tau_M^{(2)} = \tau_M(A) = \tau_M(B)$

$$\begin{tikzcd} \mathbb{R} \times T(M) \arrow[r, "\mu"] \arrow[d] & T(M) \arrow[d, "\tau_M"] \\ M \arrow[r, "="] & M \end{tikzcd}$$

$$(x, A) \mapsto \tau_M(A)$$

such that for each $x \in M$, the maps α & μ define an n-dim vector space structure on

$T_x(M) = \tau_M^{-1}[\{x\}]$, the <u>tangent space</u> <u>to</u> x <u>in</u> M.

We also want $T(f) : T(M) \longrightarrow T(N)$ to map $T_x(M)$ to $T_{f(x)}(N)$ linearly (call this restricted/corestricted map $T_x(f)$ or something similar).

Furthermore, if $U \subseteq M$ is inclusion of an open subset, we want $T(U) \rightarrow T(M)$ to be given by the inclusion of $\tau_M^{-1}[U]$ in $T(M)$, and if U is open in $\mathbb{R}^n$ we want $T(U) = U \times \mathbb{R}^n$ as above, and likewise for $T(f)$ if $f : U \rightarrow V$ is smooth.

The preceding essentially characterize the construction uniquely. The document <u>amalgamation.pdf</u> gives the often messy details of how one can construct $T(M)$ and $T(f)$. Just about everything one needs to know at this point can be summarized in the following:

STANDARD ATLASES FOR TANGENT SPACES.

Let $\mathfrak{A}$ be a smooth atlas for M, with charts $h_\alpha : \underset{\text{open in } M}{W_\alpha} \longrightarrow \underset{\text{open in } \mathbb{R}^n}{U_\alpha}$. Then there is an associated atlas of smooth charts on $T(M)$ having the form

$$H_\alpha : \underset{\tau^{-1}[W_\alpha]}{\underset{\shortparallel}{T(W_\alpha)}} \longrightarrow \underset{U_\alpha \times \mathbb{R}^n}{\underset{\cong}{T(U_\alpha)}}$$

such that the transition maps

$$\text{“}H_\beta \circ H_\alpha^{-1}\text{”} : U_{\beta\alpha} \times \mathbb{R}^n \longrightarrow U_{\alpha\beta} \times \mathbb{R}^n$$

have the form $(x, Y) \longrightarrow (\text{“}h_\beta h_\alpha^{-1}\text{”}(x), D\text{“}h_\beta h_\alpha^{-1}\text{”}(x)Y)$.

This is not quite proven in amalgamation.pdf but it can be viewed as an "exercise." [Show that if $\mathfrak{A}$ is any atlas and $\mathfrak{A}^*$ is the maximal atlas containing $\mathfrak{A}$, then the constructions in the document for $\mathfrak{A}$ and $\mathfrak{A}^*$ yield diffeomorphic objects].

We can also describe $T(f)$ locally:

Let $f: M \to N$ be smooth, let $\mathcal{B}$ be a smooth atlas for N, and let $\mathcal{A}$ be a smooth atlas for M such that each chart in $\mathcal{A}$ maps into a chart in $\mathcal{B}$ under f:

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "h_\alpha"'] & N \arrow[d, "k_\beta"] \\ U_\alpha \arrow[r, "\text{“}f\text{”}"] & V_\beta \end{tikzcd}$$

f maps $h_\alpha^{-1}[U_\alpha]$ into some $k_\beta^{-1}[V_\beta]$

Consider the corresponding situation in tangent spaces

$$\begin{tikzcd} T(M) \arrow[r, "T(f)"] \arrow[d, "H_\alpha"'] & T(N) \arrow[d, "K_\beta"] \\ U_\alpha \times \mathbb{R}^m \arrow[r, "\text{“}T(f)\text{”}"] & V_\beta \times \mathbb{R}^n \end{tikzcd}$$

The naturality properties of $\tau_M : T(M) \to M$ imply $T(f)$ maps $H_\alpha^{-1}[\ldots]$ to $K_\beta^{-1}[\ ]$ and hence

$$\text{“}T(f)\text{”}(x, Y) = (\text{“}f\text{”}(x), D\text{“}f\text{”}(x)Y).$$