4. Mersions and embeddings

Smooth mappings of maximum rank.

U open in $\mathbb{R}^n$, V open in $\mathbb{R}^m$, $f : U \to V$ smooth. Then for each $x \in U$, rank $Df(x) \leq \min(m, n)$. If $n \leq m$, equality means $Df(x)$ is 1-1. If $n \geq m$, equality means $Df(x)$ is onto.

OPENNESS PHENOMENON If $Df(x)$ has maximum rank, then there is some open neighborhood W of x s.t. $y \in W \Rightarrow Df(y)$ has max rank.

Proof Let $r = \min(m, n)$, so rank $Df(x) = r$ and rank $Df(y) \leq r$ if $y \in U$. By assumption there is an $r \times r$ submatrix A of $Df(x)$ such that $\det A \neq 0$. Let $A(y)$ be the corresponding $r \times r$ submatrix of $Df(y)$ [so $A = A(x)$]. By continuity of the determinant, there is some open neighborhood W such that $\det A(y) \neq 0$ on W. Therefore $r \leq \operatorname{rank} Df(y)$ for $y \in W$; since we already have the reverse inequality, rank $Df = r$ on W. ∎

Definitions $f : M^m \longrightarrow N^n$ smooth is

① a <u>submersion</u> if for all $x \in M$, $T_x(f)$ is onto
($\Rightarrow m \geq n$)

② an <u>immersion</u> if for all $x \in M$, $T_x(f)$ is 1-1
($\Rightarrow m \leq n$)

[If $m = n$, immersion $\Longleftrightarrow$ submersion.]

Simple examples $U \subseteq \mathbb{R}^p$, $V \subseteq \mathbb{R}^q$ open
$0 \in V$.

submersion	immersion
$U \times V \longrightarrow U$	$U \longrightarrow U \times V$
$(u, v) \longrightarrow u$	$u \longrightarrow (u, 0)$

CLASSICAL EXAMPLES OF IMMERSIONS.

Regular smooth curve: $\gamma : (a, b) \longrightarrow \mathbb{R}^n$

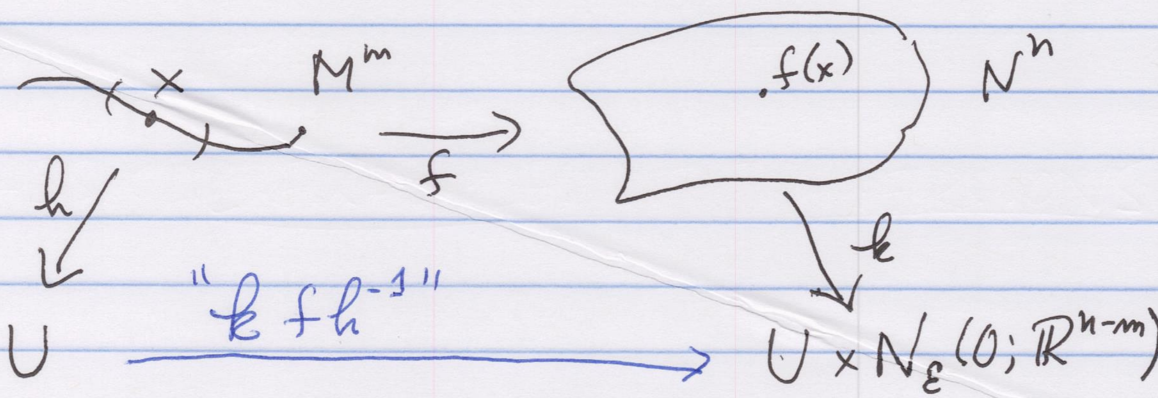
$\gamma'(t) \neq 0$ all t.

Regular surface patch: U open in $\mathbb{R}^2$;
$\sigma : U \longrightarrow \mathbb{R}^3$ smooth with $\underbrace{\frac{\partial \sigma}{\partial u} \times \frac{\partial \sigma}{\partial v}}_{\text{CROSS PRODUCT}} \neq 0$

Note that the 3×2 matrix $D\sigma$ has $\frac{\partial \sigma}{\partial u}$ and $\frac{\partial \sigma}{\partial v}$ as its columns. This matrix has rank 2 $\Longleftrightarrow$ the columns are linearly independent, which is true $\Longleftrightarrow$ $\frac{\partial \sigma}{\partial u} \times \frac{\partial \sigma}{\partial v} \neq 0$.

Local behavior of $\left\{\begin{matrix}\text{im-}\\ \text{sub-}\end{matrix}\right\}$mersions

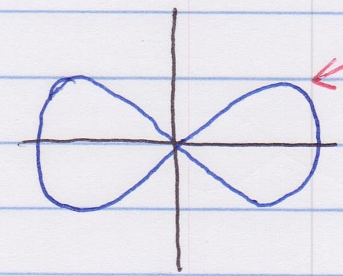
Immersions Suppose that the smooth map $f: M^m \longrightarrow N^n$ is an immersion, and let $x \in M^m$. Then there exist smooth charts



such that "kfh^{-1}"$(u) = (u, 0)$.

Corollary An immersion is locally 1-1.

EXAMPLE. The Figure 8 curve $\gamma(t) = (\sin 4\pi t, \sin 2\pi t)$ passes to a smooth immersion $g: S^1 \looparrowright \mathbb{R}^2$, but it is not globally 1-1 because $g(1) = g(-1) = 0$.



Proof of immersion statement

The main input will be the Inverse Function Thm.

Since everything in the assertion is local, it will suffice to consider immersions $f: U \to V$, where U is open in $\mathbb{R}^m$ + V is open in $\mathbb{R}^n$.

Let $L = [\text{Image } Df(x)]^{\perp}$, and let $\theta: \mathbb{R}^{n-m} \to L$ be a linear isomorphism.

Define F by

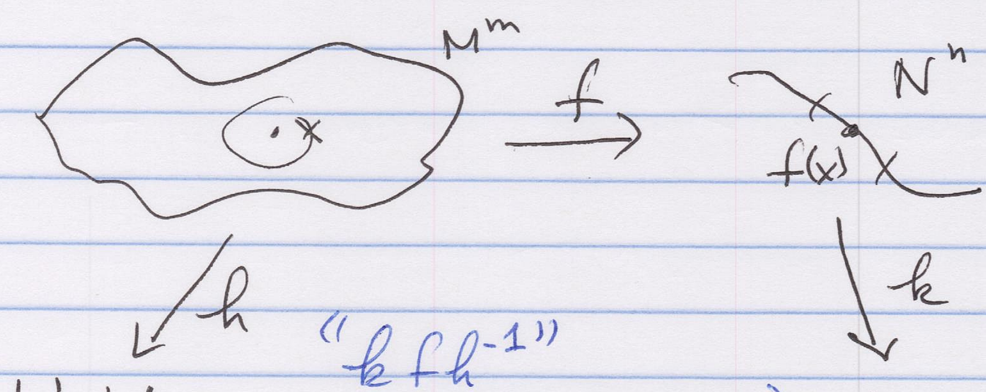
$$U \times \mathbb{R}^{n-m} \xrightarrow{1 \times \theta} U \times L \longrightarrow \mathbb{R}^n$$

$$(y, A) \longrightarrow f(y) + A.$$

Then $DF(x,0)$ is an isomorphism, so there is a local inverse $G : W \xrightarrow[\text{(into)}]{} U \times \mathbb{R}^{n-m}$.

We can choose W to be so small that $f(x) \in W \subseteq V$ if we wish to do so. and we can find $x \in U_0 \subseteq U$ so that $f[U_0] \subseteq W$. It then follows that "$G \circ f \mid U_0$" sends y to $(y, 0)$. Cutting down W further to $W_0 \cong U \times N_\varepsilon(0; \mathbb{R}^{n-m})$ yields the final conclusion. ∎

Submersions. Suppose that the smooth map $f : M^m \longrightarrow N^n$ is a submersion, and let $x \in M^m$. Then there exist smooth charts



such that "kfh^{-1}"$(u, v) = u$.

Corollary A submersion is an open mapping. [On each special chart as above the map is open, and the charts determine an open covering of M.]

Corollary If $f: M^m \longrightarrow N^n$ is a submersion with M^n and N^n connected and M^m compact, then f is onto ($\Longrightarrow N^n$ is also compact).

Proof of submersion statement

Once again, the Inverse Function Theorem is the main input, and it will suffice to consider submersions $f: U \longrightarrow V$ where U is open in $\mathbb{R}^m$ and V is open in $\mathbb{R}^n$.

One additional simplification will be useful: Since $Df(x)$ is onto, there is an invertible $m \times m$ matrix A such that $Df(x) \cdot A$ is the map sending $(t_1, \ldots, t_n)$ to $(t_1, \ldots, t_m)$ (forget the last $(n-m)$ coordinates)

By the chain rule, if $Ax_0 = x_0$ and $g(y) = f(Ay)$, then $Dg(x_0)$ is this projection onto the first n coordinates. If we can prove the result for g, then it will also be true for f by applying A^{-1} to the chart for the domain.

With this assumption, define

$$F : U \longrightarrow V \times \mathbb{R}^{m-n}$$

by f on the V factor and projection to the last $(m-n)$ coordinates on the second factor. Once again $DF(x)$ is an isomorphism, so there is a local inverse $G : V_0 \times N_\varepsilon(0; \mathbb{R}^{m-n}) \to U$. It follows immediately that the composite "$f \circ G$" sends (v, w) to v (since f = projection of F onto V). ∎