5. Smooth submanifolds

Def. Let $N \subseteq M$ be an inclusion s.t. N and M are smooth manifolds. Then N is a smooth submanifold if the inclusion $i : N \subseteq M$ is a smooth immersion ($\Rightarrow$ it is also a smooth embedding).

Before discussing submanifolds further, we need to prove a result on smooth embeddings:

LOCAL FLATNESS PROPERTY. Let $f : N^n \to M^m$ be a smooth embedding. Then for each $p \in N$ there are smooth charts $h_\alpha : \underset{\text{in } N}{W_\alpha} \to \underset{\text{in } \mathbb{R}^n}{U_\alpha}$ $(p \in W_\alpha)$

$k_\alpha : \underset{\text{in } M}{\Omega_\alpha} \to \underset{\text{in } \mathbb{R}^m = \mathbb{R}^n \times \mathbb{R}^{m-n}}{U_\alpha \times V_\alpha}$ such that

$k_\alpha f h_\alpha^{-1}(x) = (x, y_0)$ for some fixed $y_0 \in V_\alpha$ $\boxed{\text{AND}}$

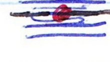
$$f[N] \cap \Omega_\alpha = k_\alpha^{-1}[U_\alpha \times \{y_0\}].$$

This is stronger than the condition for an immersion. In the latter case, some point

(w_0, y_0) might be a limit point of

$$k_\alpha^{-1}[f[N]\cap\Omega_\alpha] - U_\alpha\times V_\alpha - \{y_0\}.$$

<u>Examples</u> 1. The immersion $S^1 \longrightarrow \mathbb{R}^2$ whose image is the Figure 8 curve 

2. The immersion $\mathbb{R}\times\mathbb{N}$ ($\mathbb{N}$ = nonneg. integers) $\longrightarrow \mathbb{R}^2$ which sends $\mathbb{R}^n\times\{n\}$ to $\mathbb{R}^n\times\{q(n)\}$ 
where $q : \mathbb{N}\longrightarrow\mathbb{Q}$ is a 1-1 correspondence.

<u>**Proof.**</u> Since f is an immersion, one can find charts so that the first condition holds.

By assumption, $f h_\alpha^{-1}$ maps U_α homeomorphically to an open subset of $f[N]\subseteq M$. Therefore there is some open subset V^* in N such that $V^*\cap f[N] = f h_\alpha^{-1}[U_\alpha] = f[W_\alpha]$. Let $\Omega_\alpha^* = \Omega_\alpha\cap V^*$. Then $f[W_\alpha] = f[N]\cap\Omega_\alpha^*$, and the latter is just $k_\alpha^{-1}[U_\alpha\times\{y_0\}]$. Let $(x_0, y_0) = k_\alpha^{-1} f(p) \in k_\alpha[\Omega_\alpha^*] \subseteq U_\alpha\times V_\alpha$, and choose open neighborhoods U_α' of x_0 and V_α' of y_0 so that $U_\alpha' \times V_\alpha' \subseteq k_\alpha[\Omega^*]$. Then we have restricted charts $h_\alpha' : \longrightarrow U_\alpha'$, $k_\alpha' : \longrightarrow U_\alpha' \times V_\alpha'$ such that $p \in h_\alpha'^{-1}[U_\alpha']$, f is given locally by $u \longrightarrow (u, y_0)$ and $k_\alpha'[f[N] \cap \Omega_\alpha'] = U_\alpha' \times \{y_0\}$. $\blacksquare$

EXAMPLE Topological embeddings of manifolds need not be locally flat. The video the Alexander sphere... constructs a standard example (also see Hatcher).

COROLLARY. If $f : M^n \longrightarrow N^n$ is a smooth embedding, then $f[M]$ can be made into a smooth submanifold such that f maps M diffeomorphically onto $f[M]$.

CORESTRICTION LEMMA. Suppose f is a smooth embedding and $g : P \longrightarrow M$ is a smooth map such that $g[P] \subseteq f[N]$. Then there is a unique smooth map $g_0 : P \longrightarrow N$ such that $g = f \circ g_0$.

<u>Proof</u> Since f maps N homeomorphically onto $f[N]$, there is a unique continuous map g_0 such that $f\,g_0 = g$. We need to verify that this mapping is smooth.

Let $p \in N$ and choose locally flat charts as before:

$$\begin{tikzcd} N \arrow[r, "f"] \arrow[d, "h_\alpha"'] & M \arrow[d, "k_\alpha"] \\ V_\alpha \arrow[r, dashed] & U_\alpha \times V_\alpha \end{tikzcd}$$

$u \qquad\qquad (u, y_0)$

and no points off this slice come from $f[N]$

Suppose now that $p = g_0(p')$, and choose a smooth chart $\ell_\alpha : P \longrightarrow W_\alpha$ at p' so that

actually defined on an open subset

g_0 maps $\ell_\alpha^{-1}[W_\alpha]$ into $h_\alpha^{-1}[U_\alpha]$.

Then g maps $\ell_\alpha^{-1}[W_\alpha]$ into $k_\alpha^{-1}[U_\alpha \times V_\alpha]$ by a map which has the local form

$$w \longrightarrow (\gamma(w), y_0).$$

Since g is smooth, the map γ is also smooth. However, by construction γ is a local representation of g_0, so $g_0 \mid \ell_\alpha^{-1}[W_\alpha]$ is smooth. We can find an open covering of P by sets of the form $\ell_\alpha^{-1}[W_\alpha]$, and therefore it follows that g_0 is smooth. ∎

<u>**Proof of Corollary.**</u> By local flatness we know that $f[N]$ is contained in a union of smooth coordinate charts for M of the form $k_\alpha : \Omega_\alpha \longrightarrow U_\alpha \times V_\alpha$ such that $f[N] \cap \Omega_\alpha$ corresponds to $U_\alpha \times \{y_0\}$. Therefore we have an atlas of charts of the form $[\text{open in } f[N]] \longrightarrow U_\alpha$ for $f[N]$. By our hypotheses the transition maps

$$\begin{array}{ccc} U_\alpha \times V_\alpha & & U_\beta \times V_\beta \\ \uparrow & & \uparrow \\ \text{open subset} & \xrightarrow{\psi_{\beta\alpha}} & \text{open subset} \end{array}$$

are diffeomorphisms

and by construction the slice for $U_\alpha \times \{y_0\}$ is sent to the slice for $U_\beta \times \{y_0\}$; more precisely, $(\text{open subset})_{\beta\alpha} \cap U_\alpha \times \{y_\alpha\}$ is sent to $(\text{open subset})_{\alpha\beta} \cap U_\beta \times \{y_\beta\}$. Since $\psi_{\beta\alpha}$ is smooth (the charts for M are smooth), it follows that the restricted charts are also smooth. (Recall that if $f: U \times V \longrightarrow U' \times V'$ is smooth and sends $U \times \{y\}$ to $U' \times \{y'\}$, then the map $U \times \{y\} \longrightarrow U \times V \longrightarrow U' \times V' \longrightarrow U' \times \{y'\}$ is also smooth.)

Therefore we have a smooth atlas for $f[N]$. By construction f induces a homeo $N \longrightarrow f[N]$ and with respect to this atlas the map f is smooth such that $T_x(f)$ has maximum rank for all $x \in N$. Therefore f is a diffeomorphism by a corollary to the Inverse Function Theorem. ∎