Level sets

The preceding argument and results on submersions for Chapter 4 immediately yield the following result.

REGULAR LEVEL SET THEOREM. $n \geqslant p$

Let M^n be a smooth manifold, let $f : M^n \longrightarrow \mathbb{R}^p$ be smooth, and suppose that $0 \in \mathbb{R}^p$ is a **regular value**; i.e. $f(x) = 0$ implies that

$$T_x(f) : T_x(M) \longrightarrow \mathbb{R}^p = T_0(\mathbb{R}^p)$$

has rank p. Then $f^{-1}[\{0\}]$ is a smooth submanifold such that $\dim f^{-1}[\{0\}] = n - p$.

EXAMPLE. $f : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}$ given by

$$f(v) = |v|^2. \text{ Then } Df(v)(A) = 2\langle v, A\rangle$$

and if $|v|^2 = 1$ then $Df(v)$ has rank 1. In this case the zero set of $g(v) = f(v) - 1$ is S^n.

Now suppose that $f : S^n \longrightarrow S^n$ sends x to Px where P_x is an $(n+1) \times (n+1)$ orthogonal matrix. Its $L_P : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}^{n+1}$ is multiplication by pP, then $L_p|S^n$ is smooth because S^n is a submanifold, and by the Corestriction Lemma there is an induced smooth map $S^n \longrightarrow S^n$. Since $L_p^{-1} = L_{p^{-1}}$ it follows that this map from S^n to itself is a diffeomorphism.

Embeddings and Tangent Spaces

PROPOSITION. If $i : N \longrightarrow M$ is the inclusion of a smooth submanifold, then so is $T(i) : T(N) \longrightarrow T(M)$.

Sketch of proof. Locally the mapping i looks like $U \longrightarrow U \times V$ sending u to $(u, 0)$. (U open in $\mathbb{R}^n$, V in $\mathbb{R}^{m-n}$). Therefore $T(i)$ looks locally like the inclusion $U \times \mathbb{R}^n \longrightarrow U \times V \times \mathbb{R}^n \times \mathbb{R}^{m-n}$ sending (u, A) to $(u, 0; A, 0)$. ∎

EXAMPLE. $f:\mathbb{R}^n \longrightarrow \mathbb{R}$ smooth, 0 a regular value. Then $V = f^{-1}[\{0\}]$ is an $(n-1)$ dimensional smooth submanifold and $T(V) \subseteq T(\mathbb{R}^n) \cong \mathbb{R}^n \times \mathbb{R}^n$ is the set of all (x, y) such that $f(x) = 0$ and $\nabla f(x) \cdot y = 0$ (equivalently, $Df(x)[y] = 0$).

If $V = S^n$, f = usual function, then $T(V) = T(S^n)$ = all (x, y) such that $|x| = 1$ and $\langle x, y \rangle = 0$.

IMPORTANT POINT

If $0 \in \mathbb{R}^p$ is not in the image of $f: M^n \longrightarrow \mathbb{R}^p$, then f is automatically a regular value, for the condition

$$f(x) = 0 \Longrightarrow Df(x) \text{ is onto}$$

is vacuously true.

Another class of examples

Let $f: M \to N$ be smooth. Then the graph of f

$$\Gamma_f = \{(x,y) \in M \times N \mid y = f(x)\}$$

is a smooth submanifold of $M \times N$.

Verification. We need to show that $\gamma(x) = (x, f(x))$ is a smooth embedding.

① γ is continuous, in fact smooth, because $x \to x$ and $x \to f(x)$ are.

② γ is 1-1, for $(x, f(x)) = (x', f(x')) \Rightarrow x = x'$.

③ γ is an immersion, for $\pi_M \circ \gamma = id \Longrightarrow$

$$id = T(\pi_M) \circ T(\gamma) \Rightarrow T(\gamma) \text{ is } 1\text{-}1.$$

④ γ is closed, for Γ_f is closed in $M \times N$ and E closed in $M \Rightarrow \gamma[E] = (E \times N) \cap \Gamma_f$, an intersection of closed subsets.

Special case. f = constant map w/value y_0
$\Longrightarrow M \times \{y_0\}$ is a smooth submanifold.

Note that $\gamma[M]$ is diffeomorphic to M.

Transversality

Let $f: M \to N$ be smooth, and let $P \subseteq N$ be a smooth submanifold. Then f is transverse regular to P if for all $x \in M$ with $f(x) = p$, the composite

$$T_x(M) \xrightarrow{T_x(f)} T_{f(x)}(N) \longrightarrow T_{f(x)}(N)/T_{f(x)}(P)$$

is onto.

THEOREM. If f is transverse as above, then $f^{-1}[P] \subseteq M$ is a smooth submanifold and

$$\dim M - \dim f^{-1}[P] = \dim N - \dim P.$$

ONE TYPICAL SITUATION. Suppose that f is also a submanifold inclusion. If f is transverse to P, then $P \cap M$ is a smooth submanifold and

$$\dim (M \cap P) = \dim M + \dim P - \dim N.$$

See Lee for a proof of the theorem.